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SOME COMMENTS ABOUT DEFINITIONS

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Many students encounter a great deal of confusion in studying mathematics because they lack an understanding of the nature and role of the language used in discussing mathematics. In fact, there are different philosophical points of view as to the exact relationship between language and mathematics even between mathematicians and logicians. Nevertheless, insights are never attained unless thought and effort are given to the matter.

There are many basic things about a technical language that students should and can be informed about that will eliminate some confusion. One such enlightening endeavor is for students to give consideration to the matter of "definitions" in mathematics. The following material offers a few thoughts that can be discussed with students in order to help them appreciate the nature and purpose of definitions in mathematics.

Man communicates by the use of some language. The language (means of communication) may be in various forms of sound, written symbols, gestures, or signs. Communication takes place only when the receiver's interpretation of a message agrees with what the sender meant by the message. How can the sender of the message be assured that the receiver will interpret the message and obtain the intended meaning? If words are the means of communication, to have proper communication both the sender and receiver must somehow in advance agree on the meaning of each word. How are such agreements and understandings established?

In order to appreciate some aspects of communication by the use of words, consider an Englishman and a Frenchman. Furthermore, suppose there does not exist any English-French dictionary. If the Englishman does not understand any French, a French dictionary would be useless to him in finding the meaning of a particular word. After all, a dictionary merely indicates that a word is to have the same meaning as another word or group of words and each of these words is to have the same meaning as another group of words and so on. Furthermore, since there are only a finite number of words in any language, defining one word leads to the use of other words, each of which leads to the use of others until eventually one word will appear twice in the chain and form a loop. However, even though a dictionary defines every word by the use of other words and admits such circularity, it is assumed that the user already knows the meaning of a sufficient number of words in order to arrive at a meaning before completing such a loop. But how does one become aware of the meaning of a sufficient number of these basic words in order to use a dictionary?

To avoid meaningless circularity it is necessary to establish the meaning of some words by means other than by the mere use of other words. What means should one initially choose to convey the meaning without the need of just using other words? (Here it is assumed that some language is already established and that one is trying to make the dictionary useful.) The most easily understood words are those for which the meaning can be exhibited by physical means that a person can sense. Hence one can indicate the association between the word and the comprehendible physical object or situation which is the meaning. For instance, one can establish the meaning of "rock" by pointing with the finger to real rocks. After a long and tedious task of establishing meaning for a large number of words in this implicit manner, it is possible to give meaning to a new word in the explicit manner of just using words already defined. This pattern of development is best illustrated by observing how a child actually learns to read.

To study, develop, or apply a discipline such as mathematics, it is necessary to communicate in a special manner about mathematics. That is, it is necessary to develop a technical, mathematical language. In any logical discourse (technical language), as in any language, it is impossible for the meaning of every term (word or symbol) to be given explicitly by indicating that it has the same meaning as phrases involving other terms. There must be some terms whose meanings are given in an implicit manner. In a technical language the basic terms that are implicitly realized are called the primitive terms of the discourse. However, in the language for a mathematical system the primitive terms actually have no precise meaning other than that implicitly given by their use and presence in the postulates of the system. That the primitive terms of mathematics have no definite meaning may at first seem very strange. However, this fact will be realized as a significant and useful aspect of mathematics. It is a later assignment of meaning to these primitive terms that leads man to apply and make mathematics a useful tool. Nevertheless, the point to be made here is that a mathematical language must also have some primitive terms, but no attempt is initially made to implicitly attach a meaning to them. For this reason the primitive terms of a mathematical language are sometimes called the *undefined terms*.

All other terms in a mathematical language are either nonmathematical, nonspecialized English terms, such as "the," "as," "there," and "consider," or terms that are abbreviations for a more complicated combination of undefined terms and nonmathematical English terms. The latter terms are called the defined terms. A *defined term* is an abbreviation of a word, phrase, sentence, symbol, or any combination of such that are already part of the mathematical language. The only reason for having defined terms in the language is that they serve as a matter of convenience; otherwise, the language becomes increasingly bulky. For instance, "circle" is a shorter and more convenient word than the phrase "a set of all points in a plane equidistant from a fixed point," or the symbol "$\leqq$" is shorter than the phrase "is less than or equal to." Here "circle" is merely used as an abbreviation and is really not necessary to have in the language since one could use the already established phrase "a set of all points in a plane equidistant from a fixed point." In fact, since all defined terms of a mathematical language are merely abbreviations of an already established language, they really need never be used, since one could use in their place what they abbreviate. But a defined term used in place of what it is abbreviating or vice-versa should in no way destroy or modify any intent of a statement involving the substitution. For instance, "$2\leqq 3$" has exactly the same meaning as "2 is less than or equal to 3." In other words, a defined term and that which it replaces or abbreviates are both to reveal the same thing.

The accepted familiar general language used to develop a technical language, such as English, in this case, is called the *syntax* language. The technical language being developed, in this case a mathematical language, is referred to as the *object* language. It is also possible to develop an object language by using another object language as the syntax language, and this situation happens quite often in the field of mathematics.

It has already been mentioned that defined terms are really not necessary in mathematical (object) language, but they are used only to be economical. What mathematical expressions become worthy of abbreviation? Certainly any expression that is used quite frequently would be beneficial to abbreviate. For instance, a student does not do much work in mathematics before he realizes that the phrase "such that" is used many times. Hence it becomes convenient to introduce the defined term "$\ni$" to replace "such that." One can formulate any abbreviation for any piece of object language, but if the language is meaningless, inconsistent, or insignificant, one would find little use for it. Consequently, who would bother to formulate a defined term to abbreviate such language? Even the fact that each term in a phrase is an undefined term or an already defined term does not guarantee that the phrase is meaningful, consistent, or even the mathematical "existence" of that described by the phrase. For instance, the phrase "sum line point product difference" is meaningless. The phrase "two lines intersect and do not intersect" is inconsistent because it is self-contradictory. One would not find it very beneficial to form a defined term to abbreviate a piece of language unless it were at least meaningful and consistent.

When a piece of mathematical (object) language is used quite frequently, it is usually because the language denotes pertinent mathematical concepts. Most of such concepts revolve around either sets, variables, relations, or functions.

Consider, for instance, the idea of a variable. In any mathematical system there are usually sets whose elements have common significant mathematical properties. It becomes important to be able to refer to elements of this nature, that is, to express that one is considering any element of such a particular set. An expression in mathematical language that refers to any member of a specified set and which by agreement can be replaced by any name of an element in the set is often abbreviated by a defined term, and such a defined term is a variable. For example, "circle" in Euclidean geometry is used to abbreviate "a set of points in a plane equidistant to a fixed point in the plane." Here the understanding of the language actually refers to any set of points in the plane that satisfy the requirement of containing all points equidistant to a particular point in the plane. There are an infinite number of such sets of points in the plane, and the expression is referring to any one of them. The defined term "circle" used to abbreviate this language is then itself referring to any one of these sets and is a variable. Likewise, defined terms, such as "figure," "polygon," and "isosceles triangle," all abbreviate language that refers to any element of a particular set all of whose elements exhibit pertinent mathematical properties and are particular variables.

Consider another example within the context of the integers. The expression "an integer divisible by two" refers to any number in the set of all integers divisible by two. The expression is usually symbolized by the defined term "even number." The term "even number" then refers to any integer divisible by two and thus is a variable. Other defined terms, such as "prime number," "negative number," and "perfect number," are common defined terms that abbreviate language referring to any element in a specified set and hence are variables.

Now consider language denoting a function. This language describes a definite means of associating with every element of a particular set (domain) one and only one element of a set (range). The actual rule (association) indicated by the language that associates with every element of the domain a unique element in the range is the function. (Some would rather consider a function as a set of ordered pairs with no two pairs having the same first element. If this be the case, then consider the language denoting such a set.) Functions are fundamental in mathematics and the language indicating (naming) any one such function is usually symbolized by a defined term. Hence the defined term (symbol or word) abbreviates the language indicating the function, has the same intent as the language, and hence is also a name for the function. For example, the language "associate with every ordered pair of integers (a, b), the integer $a+(-b)$," describes an important function, and the defined term "subtraction" is used to abbreviate the phrase. Hence "subtraction" means the same thing as the phrase which describes (names) the function and is itself a name for the function. Other familiar defined terms, such as "division" or "$\div$," "differentiation," "squaring," "square root," "absolute value," and "integration," are defined terms that abbreviate language which distinguishes particular important functions and are, therefore, names for the functions.

Also in considering a function, one frequently needs to refer to the unique element in the range that the function associates with a particular element in the domain. It is common practice that if a defined term, such as "f," denotes a particular function, the "$f(b)$" is used to denote the unique element in the range that the function f associates with the element b of the domain of f. Suppose that the defined term "area" is used to abbreviate "associate with every geometric square the square of the length of a side" and thus denotes that function. The unique number associated by the function with a particular square $ABCD$ is denoted by "the square of the length of a side of the square $ABCD$," but is sometimes abbreviated by using the name "area" of the function by "the area of the square $ABCD$." Likewise, if "absolute value," "square root," etc., are defined terms denoting particular functions, then the particular element in the range associated with a given element in the domain is referred to respectively by "the absolute value of the number b," "the square root of the number b," etc. If the defined terms "$|\ |$" and "$\sqrt{\ }$" are used instead of "absolute value" and "square root," then the elements associated by the respective function with a number b are denoted by respectively "$|b|$" and "$\sqrt{b}$." There is much that could be said regarding the various languages used regarding functions, but the point to be made here is that much of such language used is abbreviated with defined terms (words or symbols), and, to communicate with such defined terms, one must know the particular language that is abbreviated and have an understanding of that language.

The notion of relations in mathematics is also of such significance, that is, discussed frequently enough, that much of the language pertaining to relations is symbolized. For instance, "parallel" or "$\|$," "similar" or "$\sim$," and "less than" or "$<$," are defined terms that abbreviate language revealing certain relations.

In summary, one has the privilege of using a defined term to replace any piece of object language concerning a mathematical system, but one usually only considers defined terms to abbreviate meaningful and consistent language that is used quite frequently. Such language is usually concerned with pertinent mathematical notions, such as sets, variables, functions, or relations.

How are the defined terms (symbols or words) chosen? The abbreviations (defined terms) actually can be quite arbitrary, but sometimes they are made to be suggestive either of the language they are abbreviating (replacing) or the meanings of such language. For instance, the defined term "equilateral triangle" suggests the language it abbreviates, namely, "a triangle with three equal sides." But, "%" with a little imagination looks like "1/100," which means the same thing as "percent"; thus "%" was chosen to abbreviate "percent" because it looks like something that has the same meaning as "percent." Consider "triangle" as the choice to abbreviate "the union of three line segments determined by three noncollinear points." Here "triangle" suggests three angles, which though not the language it abbreviates, yet is related to the mathematical meaning of the language. Even though the choice of defined terms is intended to be helpful, they sometimes can be misleading. For instance, "real number" may suggest something physical and might be misleading. One should never consider the suggestion realized by the nature of a defined term as the exact language abbreviated by the term or the complete meaning of the language. However, one often can use the suggestion made by the defined terms as an aid to recall the actual agreement made as to the language it is to replace.

An important point to be made about terms in any language and especially a mathematical (object) language is that the terms have only the meaning that man chooses to assign them. Terms are not natural physical constants with fixed meanings, but they are man-made symbols with assigned meanings established by agreements between men. The only laws or rules in such freedom comes from man himself in that the language is to serve to make ourselves understood.

In order to use defined terms unambiguously it is necessary to indicate clearly and exactly what each defined term abbreviates and replaces. This message is indicated by a statement called the definition of the defined term. A *definition* is a statement which reveals a defined term (words or symbols) and a combination of previously defined, undefined, and/or specialized English terms that it is to replace (abbreviate). The definition shows that two groups of words, phrases, sentences, and/or symbols are to have the same meaning or understanding gained from the language. A definition is merely a statement made by the sender to indicate to the receiver that a particular symbol (defined term) is introduced, the already familiar language it is to abbreviate, and that the defined term and what it abbreviates are to be used as showing the same meaning or intent.

Definitions appear in existing mathematical literature in many forms. One must be careful to notice that some statements are indeed actually definitions. Consider for example some common forms that might be meant to serve as mathematical definitions where in all cases the defined terms appear on the left and the already familiar object language that it is to abbreviate and of which it has the same meaning is on the right.

Addition is commutative *means* for every number x and y, $x+y=y+x$.
$|x|=\sqrt{x^2}$.
$x-y$ *is defined* as $x+(-y)$.
$\varnothing$ *denotes* the empty set.
$\forall$ *is notation for* "for all".
"$\|$" *is read* "parallel to".

Between each defined term and that which it abbreviates appears an expression indicating that one is to be symbolic of the other and have the same meaning. It might be helpful if definitions appeared in a more uniform manner instead of in so many various ways. If a symbol, such as "$=df$" (read "is equal by definition") were used to denote a definition, then the above examples would appear in a somewhat more uniform manner as:

"Addition is commutative" $=df$ "for every number x and y, $x+y=y+x$".
"$|x|$" $=df$ "$\sqrt{x^2}$".
"$x-y$" $=df$ "$x+(-y)$".
"$\varnothing$" $=df$ "the empty set".
"$\forall$" $=df$ "for all".
"$\|$" $=df$ "parallel to".

Mathematical definitions as introduced here are statements (agreements) to be made about the language of mathematics and not particularly about the mathematics. In other words, definitions are concerned with what is used in discussing mathematical entities and not the entities themselves. In fact, definitions are not even needed to discuss mathematics. A definition indicates that a defined term is to abbreviate and mean the same thing as a particular piece of language all of whose terms are already part of the object language. One really need not have or use the defined term, but instead he may use the already familiar language it abbreviates. Hence, there need not be a definition relative to the defined term if one really does not need to use the term. In other words, an abbreviation of language need not be used; one can always use that which is abbreviated.

It is also common practice for "definitions" to be presented in biconditional (p if and only if q) forms. To introduce a defined term by the use of an expression in such a biconditional form requires some additional agreements that are not inherent from just the connective function "if and only if." These agreements are not always appreciated or may not be fully understood.

In the first place, the connective "if and only if," or simply "iff," is itself a function that associates with every pair of statements p and q another statement "p iff q," whose truth value is "true" only when the truth values of p and q are the same. Now suppose the connective "iff" is used between two statement functions $s(x)$ and $t(x)$, each, say in a variable x, as "$s(x)$ iff $t(x)$." Here "$s(x)$ iff $t(x)$" is also a statement function only because, if the variable x in "$s(x)$ iff $t(x)$" is replaced by a constant b, both $s(x)$ and $t(x)$ become statements $s(b)$ and $t(b)$ and, therefore, "$s(b)$ iff $t(b)$" is a statement by definition of "iff." If either $s(b)$ or $t(b)$ were not statements, then "$s(b)$ iff $t(b)$" would be meaningless, since "iff" is defined only on ordered pairs of statements.

Now, suppose one is to introduce a defined term as "even" *only* by the expression "x is an even integer iff x is divisible by 2," with no additional agreements. If the variable x in such an expression were replaced by a constant, say b, then the expression becomes "b is an even integer iff b is divisible by 2." Now "b is divisible by 2" can be verified from the mathematical system of integers as "true" or "false" and, thus, is a statement. But "b is an even integer" by itself is not a statement since "even" is not yet a term of the system. The connective function "iff" is not defined on this pair of expressions since one, namely, "b is an even integer," is not a statement. It follows then that "x is an even integer iff x is divisible by 2" is not even a statement function since any replacement of x does not yield a statement. Thus the entire expression is useless. However, if the expression, "x is an even integer iff x is divisible by 2," is said to be a definition, then there are additional agreements made which makes the expression meaningful. To state that such an expression is a definition, it is first understood that "even" is to be a term for which "x is an even integer" is a statement function. Furthermore, it is understood that the biconditional is to be an equivalence; that is, "x is an even integer iff x is divisible by 2" is to be "true" for any replacement of the variable x. Hence "x is an even integer" and "x is divisible by 2" for the same replacements of x are to yield statements of the same truth value.

At this point, it is known that "even" is to be a term, such that to state "b is an even integer" is a statement and is either "true" or "false" depending on whether "b is divisible by 2" is verified as "true" or "false" in the system. This does not specify an already familiar piece of language that "even" is to abbreviate, but does in the long run furnish a characterization criteria for labeling "true" or "false" to expressions involving an integer's being even. To introduce a term in a mathematical system in such a manner as to clearly indicate its role in the "truth" or "falsity" of statements involving this term and, in turn, to use the term instead of other characterizing statements (substitution of equivalent statements) without changing any "truth" or "falsity" is the most one obtains from any defined or undefined term in a mathematical system. After all, the essence of the game of mathematics is the labeling of "true" or "false" to certain statements because of a logical consequence of labeling the postulates as "true." Therefore, the introduction of a term in this manner is sufficient for the use of the term in mathematical endeavor. However, it is also sufficient for mathematical purposes to consider a defined term as just an abbreviation of a given piece of already familiar language, as previously indicated. The latter seems to this author as the less confusing and is also sufficient to accomplish the same end result. For instance, the undefined term "even" could be introduced simply by

"even integer" $=df$ "an integer divisible by 2".

To consider *all* defined terms (notation, words, etc.) as abbreviations, with a definition as the statement indicating the abbreviation and the language it is to abbreviate, would seem to be a more general and simple approach to the idea of definitions.

There are certain unwritten rules in the use of mathematical terms and definitions. During the course of development of mathematical language, certain words and symbols have been accepted in a particular mathematical context to have somewhat standard definitions. That is, the same term is used by all to replace somewhat the same language. If a mathematician uses such a familiar term in the common mainstream manner, then it is not always necessary to exhibit the definition. However, it is one's right to use any term in any manner as long as communication is not sacrificed. If one chooses to use a familiar term but in a different manner than would be expected, a course of action which is permissible though not advisable, then the person is obligated for the sake of clarity to explicitly reveal a definition.

Another existing situation, possibly unfortunate, is that certain terms, especially symbols, have several standard definitions. To know which definition to consider depends on the context in which the term is used and left for the judgment of the reader. For example, "$-$" may appear as part of a name for a negative number as "-2"; to indicate the image (difference) of two numbers under the binary operation (function) subtraction as "$6-3$"; to indicate the additive inverse (opposite) of a number as "$-(+2)$"; or the difference of sets as "$A-B$." However, when "$-$" appears in a piece of language it is assumed that the reader would consider the definition for "$-$" which makes the language meaningful. As in the previous examples, it would be clear that the "$-$" in "$-(+2)$" does not mean difference, for there is no indication of a minuend. But there are situations that can be ambiguous. For instance, if the convention is used where "5" abbreviates "$+5$," then the language "-5" is meaningful to consider the "$-$" as either part of the standard numeral for the negative number or as indicating the additive inverse of the number $+5$. In this case, however, and I might add "thank heavens," either meaningful interpretation refers to the same number. That is, negative five is the additive inverse of positive five. Ambiguity usually leads to more disastrous results.

A NOTE ON $N!$

JOHN E. MAXFIELD, Kansas State University

In this note two theorems are proved. The first states:

THEOREM 1. *If A is any positive integer having m digits, there exists a positive integer N such that the first m digits of $N!$ constitute the integer A.*

In order to prove this theorem the following two lemmas are needed.

LEMMA 1. *The fractional part of* $\log N$, *written* $\{\log N\}$ *is dense on the unit interval.*

To consider *all* defined terms (notation, words, etc.) as abbreviations, with a definition as the statement indicating the abbreviation and the language it is to abbreviate, would seem to be a more general and simple approach to the idea of definitions.

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LEMMA 1. *The fractional part of* $\log N$, *written* $\{\log N\}$ *is dense on the unit interval.*

Proof. $\{\text{Log}_{10} x\}$ determines all of the digits of x and the integral part determines only the location of the decimal point. The terminating decimals are dense in $[1, 10]$ and $\log x = \{\log_{10} x\}$, $1 \leqq x \leqq 10$ is continuous. Therefore $\{\log_{10} x\}$ is dense on $[0, 1]$. Since $\log_a(x) = \log_b(x)\log_a b$, the lemma follows for any base.

LEMMA 2. *$\{\log N!\}$ is dense on $[0, 1]$.*

Proof.

$$\log(N+k)! = \log N! + \sum_{j=1}^{k} \log\left(1+\frac{j}{N}\right)N$$

$$= \log N! + k\log N + \sum_{j=1}^{k} \log\left(1+\frac{j}{N}\right)$$

$$= \log N! + k\log N + \sum_{j=1}^{k}\left(\frac{j}{N} + O\left(\frac{j^2}{2N^2}\right)\right)$$

$$= \log N! + k\log N + \frac{k}{2N}(k+1) + O\left(\frac{k^4}{N^2}\right).$$

We shall now construct an η-net on $[0, 1]$ with elements of $\{\log N!\}$ where η is small.

Let $\eta > 0$. Since $\{\log N\}$ is dense on $[0, 1]$, there is an infinite set of N, say N_α, such that $5\eta/8 < \{\log N_\alpha\} < 7\eta/8$. Let $M-1 = [16/9\eta]$, the greatest integer in $16/9\eta$. Choose N_α so large that

$$\left|\frac{M(M+1)}{2N_\alpha} + O\left(\frac{M^4}{N_\alpha^2}\right)\right| < \frac{\eta}{16}.$$

Then

$$\{\log(N_\alpha + k)!\} = \left\{\log N_\alpha! + k\log N_\alpha + \frac{k(k+1)}{2N_\alpha} + O\left(\frac{k^4}{N_\alpha^2}\right)\right\} = g(k).$$

Since N_α is fixed, $\log N_\alpha$ and $\log N_\alpha!$ are fixed. For $k \leqq M$

$$\left|\frac{k(k+1)}{2N_\alpha} + O\left(\frac{k^4}{N_\alpha^2}\right)\right| < \frac{\eta}{16}.$$

Therefore the principal part of the variation of $g(k)$ as k varies is due to the term $k\log N_\alpha$. If $g(k) < 1-\eta$, and since $\{\log N_\alpha\} < 7\eta/8$, then

$$g(k+1) - g(k) < \{\log N_\alpha\} + 2\left|\frac{M(M+1)}{2N_\alpha} + O\left(\frac{M^4}{N_\alpha^2}\right)\right|$$
$$< 7\eta/8 + \eta/8 = \eta.$$

If $g(k) > 1-(1/2)\eta$, since $\{\log(N_\alpha)\} > (5/8)\eta$, it follows that $g(k+1) > (7/8)\eta + (1/8)\eta = \eta$. Since $M\{\log N_\alpha\} > (16/9\eta + 1)(5/8)\eta > (16/9\eta)(5/8)\eta = 10/9$, the whole unit interval is covered by an η net of points of $\{\log N!\}$ where the chosen N are $N_\alpha, N_\alpha+1 \cdots N_\alpha+M$.

Proof of Theorem 1. If we take log to the base 10, we want there to exist an integer t and an integer N such that $A\times 10^t \leqq N! < (A+1)10^t$ or that $t+\log A \leqq \log N! < t+\log(A+1)$. This can be done if

$$\{\log A\} < \{\log N!\} < \{\log(A+1)\}.$$

By Lemma 2 $\{\log N!\}$ is dense in $[0, 1]$. Thus, such an N can be chosen and the theorem is proved.

Theorem 2 is a generalization of Lemma 2. We need the following definitions and lemma.

Define $\log^{(j)} m$ *to be the jth iterant of* $\log m$, *i.e.*, $\log^{(2)} m = \log\log m$.

Define $\eta!^{(j)}$ *to be the jth iterant of* $n!$, *i.e.*, $n!^{(2)} = n!!$

LEMMA 3. *For* $j > 1$,

$$\log^{(j)}(n+k)!^{(j)} = \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right).$$

Proof. By Stirling's formula,

$$\begin{aligned}\log^{(2)}(n+k)!^{(2)} &= \log([(n+k)! + 1/2]\log(n+k)! - (n+k)!\log e + O(1))\\ &= \log[(n+k)!\log(n+k)!]\left[1 + O\left(\frac{1}{\log(n+k)!}\right)\right]\\ &= \log(n+k)! + \log\log(n+k)! + O\left(\frac{1}{\log(n+k)!}\right).\end{aligned}$$

From the proof of Lemma 2, $\log^{(2)}(n+k)!^{(2)}$

$$\begin{aligned}&= \log n! + k\log n + \log[\log n! + k\log n] + O(k^2/n^2)\\ &= \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right),\end{aligned}$$

establishing a basis.

Induction hypothesis:

$$\log^{(j)}(n+k)!^{(j)} = \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right).$$

Then by Stirling's formula,

$$\begin{aligned}&\log^{(j+1)}(n+k)!^{(j+1)}\\ &\quad= \log^{(j)}([(n+k)!^{(j)} + 1/2]\log(n+k)!^{(j)} - (n+k)!^{(j)}\log e + O(1))\\ &\quad= \log^{(j)}\left[(n+k)!^{(j)}\left[1 + 1/2\,\frac{\log(n+k)!^{(j)}}{(n+k)!^{(j)}} + O\left(\frac{1}{(n+k)!^{(j)}}\right)\right]\right]\end{aligned}$$

$$= \log^{(j-1)}\left(\log(n + k)!^{(j)} + 1/2\frac{\log(n + k)!^{(j)}}{(n + k)!^{(j)}} + O\left(\left(\frac{\log(n + k)!^{(j)}}{(n + k)!^{(j)}}\right)^2\right)\right)$$

$$= \log^{(j-1)}\left[\log(n + j)!^{(j)}\left[1 + 1/2\frac{1}{(n + k)!^{(j)}} + O\left(\frac{\log(n + k)!^{(j)}}{[(n + k)!^{(j)}]^3}\right)\right]\right]$$

$$= \log^{(j)}(n + k)!^{(j)} + O\left(\frac{1}{(n + k)!^{(j)}}\right)$$

$$= \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right),$$

completing the proof.

From this, as in Lemma 2, one can construct an η net and prove:

THEOREM 2. $\{\log^{(j)} N!^{(j)}\}$ *is dense on the unit interval.*

Supported in part by NSF Grant NSF GP 5938.

A CATEGORICAL SYSTEM OF AXIOMS FOR THE COMPLEX NUMBERS

W. BOSCH and P. KRAJKIEWICZ, University of Nebraska

In many elementary analysis texts the real numbers are introduced by means of a categorical system of axioms. In contrast one finds in complex analysis texts that the complex numbers are defined by means of some constructive process. The purpose of this note is to remedy this omission by giving a brief exposition of the complex numbers starting from a categorical system of axioms. In this paper it is assumed that the elementary properties of a complete linearly ordered field are known.

DEFINITION 1. *A complex number system K is defined to be any commutative field K with the following property: There exists a homomorphism $T: K \to K$ such that*

(1) $T(z) \neq z$ *for at least one* $z \in K$,

(2) $T(T(z)) = z$ *for all* $z \in K$,

(3) *the subset* $R = \{z \in K: T(z) = z\}$ *is a complete linearly ordered field.*

The mapping T is called the conjugate operator. For any element $z \in K$, $T(z)$ is called the conjugate of z and we denote it by $\bar{z}$. The elements z in K are called complex numbers.

THEOREM 1. *For any complex number system K, the following properties of the conjugate operator are valid:*

(4) $\bar{z} \neq z$, *for at least one* $z \in K$,

(5) $\bar{\bar{z}} = z$, *for all* $z \in K$,

$$= \log^{(j-1)}\left(\log(n+k)!^{(j)} + 1/2\,\frac{\log(n+k)!^{(j)}}{(n+k)!^{(j)}} + O\left(\left(\frac{\log(n+k)!^{(j)}}{(n+k)!^{(j)}}\right)^2\right)\right)$$

$$= \log^{(j-1)}\left[\log(n+j)!^{(j)}\left[1 + 1/2\,\frac{1}{(n+k)!^{(j)}} + O\left(\frac{\log(n+k)!^{(j)}}{[(n+k)!^{(j)}]^3}\right)\right]\right]$$

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(6) $$\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2, \quad \textit{for all } z_1, z_2 \in K,$$

(7) $$\overline{z_1 z_2}=\bar{z}_1\bar{z}_2, \quad \textit{for all } z_1, z_2 \in K,$$

(8) $$\bar{z}=0 \quad \textit{iff } z=0,$$

(9) $$\bar{1}=1,$$

(10) $$\overline{-z}=-\bar{z}, \quad \textit{for all } z \in K,$$

(11) $$\overline{z_1-z_2}=\bar{z}_1-\bar{z}_2, \quad \textit{for all } z_1, z_2 \in K,$$

(12) $$\overline{(z_1/z_2)}=\bar{z}_1/\bar{z}_2, \quad \textit{for all } z_1, z_2 \in K, z_2 \neq 0.$$

Proof. Properties (4)–(7) follow directly from Definition 1. To establish property (8), we note that

$$z+\bar{0}=\bar{\bar{z}}+\bar{0}=\overline{\bar{z}+0}=\bar{\bar{z}}=z \quad \text{for all } z \in K.$$

Thus $\bar{0}=0$.

To prove property (9), we observe that $z\bar{1}=\bar{\bar{z}}=\overline{\bar{z}1}=\overline{\bar{z}1}=z$, for all $z\in K$. Hence $\bar{1}=1$. The proofs of properties (10)–(12) are also easily established. This completes the proof of Theorem 1.

DEFINITION 2. *A complex number z is called skew conjugate or pure imaginary iff $\bar{z}=-z$. A complex number z is called real iff $\bar{z}=z$, that is $z\in R$.*

THEOREM 2. *If z is any complex number then $z+\bar{z}$ and $z\bar{z}$ are real. Also $z-\bar{z}$ is pure imaginary. In particular 1 is real. Also 0 is both real and pure imaginary. Finally a complex number z is both real and pure imaginary iff $z=0$.*

THEOREM 3. *In any complex number system K, the equation*

(13) $$z^2=-1$$

has two distinct solutions. If i denotes one solution, then $-i$ is the other solution.

Note that $\bar{i}\neq 0$ and that $\bar{i}=-i$. Thus i is not real and is pure imaginary.

Proof. In order to show the existence of a complex number satisfying equation (13), we proceed as follows: First, by equation (4), there is some complex number z_0 such that $\bar{z}_0\neq z_0$. Let $W=z_0-\bar{z}_0$. Observe that $W\neq 0$ and $\overline{W}=-W$. Thus W is not real but is pure imaginary. Now let $N=W\overline{W}$. Observe that $N\neq 0$ and that N is real. Hence $W^2=-N$. If $N<0$, then $W=\pm(-N)^{1/2}$ is real which is impossible. Thus $N>0$. Let $i=W/(N)^{1/2}$. Then $i^2=-1$. This completes the proof of the theorem.

THEOREM 4. *Any complex number z can be written uniquely in the form*

(14) $$z=x+iy,$$

where x and y are real and i is a fixed complex number such that $i^2=-1$.

Proof of uniqueness. Let $z=x_1+iy_1=x_2+iy_2$ where x_1, x_2, y_1, y_2 are real. Then we also have $\bar{z}=x_1-iy_1=x_2-iy_2$ and from these two equations we get $x_1=x_2$ and $y_1=y_2$.

Proof of existence. To establish existence, we observe

$$z = x + iy \tag{15}$$

where

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}. \tag{16}$$

Evidently x and y are real. This completes the proof of the theorem.

THEOREM 5. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where x_1, x_2, y_1, y_2 are real, then the following are valid:*

$$z_1 = z_2 \text{ iff } x_1 = x_2 \quad \text{and} \quad y_1 = y_2, \tag{17}$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \tag{18}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1), \tag{19}$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2), \tag{20}$$

$$z_1/z_2 = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2}, \quad \text{if } z_2 \neq 0, \tag{21}$$

$$\bar{z}_1 = x_1 - iy_1, \tag{22}$$

$$z_1 \bar{z}_1 = x_1^2 + y_1^2, \tag{23}$$

$$z_1 \text{ is real iff } y = 0, \text{ and } z_1 \text{ is pure imaginary iff } x_1 = 0. \tag{24}$$

THEOREM 6. *Any two complex number systems are isomorphic.*

Proof. Let K_1 and K_2 be any two complex number systems with respective conjugate operators T_1 and T_2. Let

$$R_1 = \{z \in K_1 \colon T_1(z) = z\} \quad \text{and} \quad R_2 = \{z \in K_2 \colon T_2(z) = z\}.$$

Now R_1 and R_2 are two complete linearly ordered fields. Thus R_1 and R_2 are isomorphic. Let $f\colon R_1 \to R_2$ be the isomorphism. Finally let i_1 be a fixed complex number in K_1 such that $i_1^2 = -1$ and let i_2 be a fixed complex number in K_2 such that $i_2^2 = -1$. We now define $F\colon K_1 \to K_2$ by

$$F(z) = F(x + i_1 y) = f(x) + i_2 f(y), \tag{25}$$

for all $z = x + i_1 y$ in K_1 where $x, y \in R_1$. It is not difficult to show that F is an isomorphism of K_1 onto K_2. This completes the proof of the theorem.

THEOREM 7. *There exists at least one complex number system.*

Proof. Let R be any complete linearly ordered field. Let $K = \{(x, y) \colon x, y \in R\}$ and define the binary operations $+$ and $\cdot$ on K by the conditions that

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2), \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \end{aligned} \tag{26}$$

Evidently K is a commutative field. Now define $T\colon K \to K$ by

$$T[(x, y)] = (x, -y), \tag{27}$$

for all $(x, y) \in K$. Clearly T is a homomorphism which satisfies properties (1), (2), and (3). Hence K is a complex number system. This completes the proof of the theorem.

A PROPERTY OF THIRD ORDER GNOMON-MAGIC SQUARES

CHARLES W. TRIGG, San Diego, California

A 3-by-3 array may be sectioned into a 2-by-2 array and a 5-element gnomon in four ways.

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$$

If the sums of the elements in the four 2-by-2 arrays are equal, then the sums of the elements in the four gnomons are equal also. Thus such a third order square can be tersely designated as *gnomon-magic*, even though the magic constant S, dealt with here, is the sum of the elements in a second order subsquare.

Since $S = a_1 + b_1 + a_2 + b_2 = b_1 + c_1 + b_2 + c_2$, then

$$a_1 + a_2 = c_1 + c_2.$$

Similarly,

$$a_2 + a_3 = c_2 + c_3$$

so

$$a_1 - a_3 = c_1 - c_3$$

or

$$a_1 - c_1 = a_3 - c_3 = c_2 - a_2.$$

Also, $b_1 + c_1 = b_3 + c_3$, so $b_3 - b_1 = -(c_3 - c_1)$.

The value of the determinant of the gnomon-magic square is

$$D = \begin{vmatrix} a_1 - c_1 & b_1 & c_1 \\ a_2 - c_2 & b_2 & c_2 \\ a_3 - c_3 & b_3 & c_3 \end{vmatrix} = (a_1 - c_1) \begin{vmatrix} 1 & b_1 & c_1 \\ -1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} = (a_1 - c_1) \begin{vmatrix} b_1 + b_2 & c_1 + c_2 \\ b_3 - b_1 & c_3 - c_1 \end{vmatrix}$$

$$= (a_1 - c_1)(c_3 - c_1)(b_1 + b_2 + c_1 + c_2) = (a_1 - c_1)(c_3 - c_1)S.$$

Consequently, S divides D.

$$T[(x, y)] = (x, -y), \tag{27}$$

for all $(x, y) \in K$. Clearly T is a homomorphism which satisfies properties (1), (2), and (3). Hence K is a complex number system. This completes the proof of the theorem.

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Consequently, S divides D.

THE CIRCUMRADIUS OF A SIMPLEX

VLADIMIR F. IVANOFF, San Carlos, California

The method of determining the value of the circumradius of a simplex in terms of the lengths of its edges, proposed in this paper, is so simple that the present writer does not exclude the possibility that it was discovered shortly after the determinants had found their place in mathematics.

Let R_n be the length of the circumradius of the n-dimensional simplex, with the vertices $A_1, A_2, \cdots, A_{n+1}$, and circumcenter at a point P. If we draw the radii from P to the vertices of the simplex, the total new configuration may be interpreted as a degenerate $(n+1)$-simplex, with zero $(n+1)$-volume (content). The n-volume, V_n, of the n-simplex, is well known [1]. If we denote the edges A_iA_j by a_{ij}, or simply by (ij), when there is no ambiguity, we have:

$$(1) \qquad (-1)^{n+1}2^n(n!)^2V_n^2 = \begin{vmatrix} 0 & 1 & 1 & \cdots 1 & 1 \\ 1 & 0 & a_{1,2}^2 & \cdots a_{1,n}^2 & a_{1,n+1}^2 \\ 1 & a_{2,1}^2 & 0 & \cdots a_{2,n}^2 & a_{2,n+1}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n,1}^2 & a_{n,2}^2 & \cdots 0 & a_{n,n+1}^2 \\ 1 & a_{n+1,1}^2 & a_{n+1,2}^2 & \cdots a_{n+1,n}^2 & 0 \end{vmatrix};$$

and for our $(n+1)$-simplex we have:

$$0 = \begin{vmatrix} 0 & 1 & 1 & \cdots 1 & 1 \\ 1 & 0 & a_{1,2}^2 & \cdots a_{1,n+1}^2 & R_n^2 \\ 1 & a_{2,1}^2 & 0 & \cdots a_{2,n+1}^2 & R_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1,1}^2 & a_{n+1,2}^2 & \cdots 0 & R_n^2 \\ 1 & R_n^2 & R_n^2 & \cdots R_n^2 & 0 \end{vmatrix},$$

from which we can find the circumradius of the n-simplex.

To do this we subtract from the elements of the last row the elements of the first one, each multiplied by R_n^2, and then subtract from the elements of the last column the elements of the first one, each multiplied also by R_n^2, and we get:

$$0 = \begin{vmatrix} 0 & 1 & 1 & \cdots 1 & 1 \\ 1 & 0 & a_{1,2}^2 & \cdots a_{1,n+1}^2 & 0 \\ 1 & a_{2,1}^2 & 0 & \cdots a^2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1,1}^2 & a^2 & \cdots 0 & 0 \\ 1 & 0 & 0 & \cdots 0 & -2R_n^2 \end{vmatrix}.$$

By expansion of this determinant, using the elements of the last row (or column, if we wish) as the leaders, we get a relation, from which we find:

$$-2R_n^2 = \left|\begin{matrix} 0 & a_{1,2}^2 & \cdots & a_{1,n}^2 & a_{1,n+1}^2 \\ a_{2,1}^2 & 0 & \cdots & a_{2,n}^2 & a_{2,n+1}^2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{n,1}^2 & a_{n,2}^2 & \cdots & 0 & a_{n,n+1}^2 \\ a_{n+1,1}^2 & a_{n+1,2}^2 & \cdots & a_{n+1,n}^2 & 0 \end{matrix}\right| \Bigg/ \left|\begin{matrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & a_{1,2}^2 & \cdots & a_{1,n}^2 & a_{1,n+1} \\ 1 & a_{2,1}^2 & 0 & \cdots & a_{2,n}^2 & a_{2,n+1}^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & a_{n,1}^2 & a_{n,2}^2 & \cdots & 0 & a_{n,n+1}^2 \\ 1 & a_{n+1,1}^2 & a_{n+1,2}^2 & \cdots & a_{n+1,n}^2 & 0 \end{matrix}\right|.$$

Denoting the numerator by D_n and taking the value of the denominator from (1), we get:

$$-2R_n^2 = D_n/(-1)^{n+1}2^n(n!)^2V_n^2,$$

whence

$$R_n = \{(-1)^n D_n\}^{\frac{1}{2}}/2^{\frac{1}{2}(n+1)}n!V_n.$$

For a triangle ($n=2$) with the sides a, b and c and area Δ, we find: $D_n = 2a^2b^2c^2$ and therefore $R_n = abc/4\Delta$. For a tetrahedron ($n=3$) we have:

$$D_3 = \left|\begin{matrix} 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 \\ a_{21}^2 & 0 & a_{23}^2 & a_{24}^2 \\ a_{31}^2 & a_{32}^2 & 0 & a_{34}^2 \\ a_{41}^2 & a_{42}^2 & a_{43}^2 & 0 \end{matrix}\right| = 16S(S - a_{12}a_{34})(S - a_{13}a_{24})(S - a_{14}a_{23}),$$

which is sixteen times the square of the area Δ' of a triangle with the sides numerically equal to the products $a_{12}a_{34}$, $a_{13}a_{24}$, and $a_{14}a_{23}$, [2], and $S = \frac{1}{2}(a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})$ = half-perimeter of the triangle. Therefore $R_3 = \Delta'/6V$.

Question: What, if any, is the geometric interpretation of D, $n > 3$?

References

1. D. M. Y. Sommerville, An Introduction to the Geometry of n Dimensions, Dover, New York, 1958, p. 125.

2. Nathan Altshiller-Court, Modern Pure Solid Geometry, Macmillan, New York, p. 248.

ON DEFINING THE SINE AND COSINE

MARLOW SHOLANDER, Case Western Reserve University

If we wish to divorce trigonometry from intuition, we may define the sine and cosine by power series, by differential equations, or we may define their inverse functions by definite integrals. With these methods the circular functions are not available to illustrate the theory which precedes a theory of series, of integrals, or of differential equations.

It is not generally known that, through Fibonacci series, we have a way of introducing these functions at a lower level. The author has been unable to

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It is not generally known that, through Fibonacci series, we have a way of introducing these functions at a lower level. The author has been unable to verify a conjecture that Sierpinski has described this method, or its equivalent. The process, briefly given below, seems worthy of wider publicity.

We define $\{a_0, a_1, \cdots; b_0, b_1, \cdots\}$ as a Fibonacci trigonometric double sequence, or an FTD sequence, if

$$a_0 = 1, \quad b_0 = 0, \quad a_1^2 + b_1^2 = 1, \tag{1}$$

$$a_{n+1} = 2a_1a_n - a_{n-1}, \tag{2}$$

$$b_{n+1} = 2a_1b_n - b_{n-1}. \tag{3}$$

The following lemma, trivial for $n=0$ and $n=1$, is proved by induction.

LEMMA. *In an* FTD *sequence, we have for* $n=0, 1, \cdots$, *that*

$$a_{n+1} = a_na_1 - b_nb_1,$$
$$b_{n+1} = a_nb_1 + b_na_1.$$

After verification for $k=0$ and $k=1$, the following theorem is easily proved by induction on k.

THEOREM 1. *An* FTD *sequence is cosinusoidal over nonnegative integers, i.e., it admits the identities*

$$a_k^2 + b_k^2 = 1, \tag{4}$$

$$a_{n+k} = a_na_k - b_nb_k, \tag{5}$$

$$b_{n+k} = b_na_k + a_nb_k. \tag{6}$$

Other identities follow in ways made familiar by trigonometry.

COROLLARY. *Since an* FTD *sequence is cosinusoidal over nonnegative integers, we have for such subscripts that*

$$a_{n-k} = a_na_k + b_nb_k, \tag{6}$$

$$b_{n-k} = b_na_k - a_nb_k, \tag{7}$$

$$a_{n+k} + a_{n-k} = 2a_na_k, \tag{8}$$

$$a_{n+k} - a_{n-k} = -2b_nb_k, \tag{9}$$

$$b_{n+k} + b_{n-k} = 2b_na_k, \tag{10}$$

$$b_{n+k} - b_{n-k} = 2a_nb_k, \tag{11}$$

$$a_2 + a_4 + \cdots + a_{2n} = a_{n+1}b_n/b_1. \tag{12}$$

Given an FTD sequence $\{a_0, \cdots; b_0, \cdots\}$ we may form another, say $\{A_0, \cdots; B_0, \cdots\}$, called its bisection, as follows. Let $A_0=a_0$, $B_0=b_0$,

$$A_1 = \sqrt{\frac{1+a_1}{2}}, \quad \text{and} \quad B_1 = \sqrt{\frac{1-a_1}{2}}.$$

Let the remaining terms be defined by recursion relations (2) and (3) with small letters replaced by caps.

The next theorem is rather immediate. For example, since $A_0=a_0$ and $A_2=a_1$, we have by induction that $A_{2n+2}=2A_2A_{2n}-A_{2n-2}=2a_1a_n-a_{n-1}=a_{n+1}$.

THEOREM 2. *If* FTD *sequence* $\{a_0, \cdots; b_0, \cdots\}$ *has bisection* $\{A_0, \cdots; B_0, \cdots\}$ *then* $A_{2n}=a_n$ *and* $B_{2n}=b_n$. *If the former has (minimal) period* $p=2^m$, *so* $a_{n+p}=a_n$ *and* $b_{n+p}=b_n$, *then the latter has period* $2p$.

To avoid a profusion of subcases we now consider a special FTD sequence and its successive bisections, namely

$$S_1 = \{1, -1, 1, -1, \cdots; 0, 0, \cdots\},$$
$$S_2 = \{1, 0, -1, \cdots; 0, 1, 0, \cdots\},$$
$$S_3 = \{1, 1/\sqrt{2}, 0, \cdots; 0, 1/\sqrt{2}, 1, \cdots\}, \cdots.$$

Clearly S_n has period 2^n and, for $n>1$, a "first quadrant" identified as

$$Q_n = \{a_0, \cdots, a_q; b_0, \cdots, b_q\}$$

where $q=2^{n-2}$, $a_q=0$, and $b_q=1$.

THEOREM 3. *In the first quadrant of* S_n, a_n *decreases from* 1 *to* 0 *and* b_n *increases from* 0 *to* 1.

Proof. The case $n=2$ is trivial. We proceed by induction, assume the result for Q_n and consider $Q_{n+1}=\{A_0, \cdots, A_{2q}; B_0, \cdots, B_{2q}\}$. First, from $a_k>a_{k+1}$, we have $A_{2k}>A_{2k+2}$ and hence $B_{2k}<B_{2k+2}$. Also

$$A_{2k+1} > A_1A_{2k+1} = (A_{2k} + A_{2k+2})/2 > A_{2k+2}.$$

Finally, $A_{2k}>A_{2k+1}$, since

$$B_{2k} < (B_{2k} + B_{2k+2})/2 = A_1B_{2k+1} < B_{2k+1}.$$

We may extend S_n to T_n, a two-way FTD sequence, by defining $a_{-m}=a_m$ and $b_{-m}=-b_m$. The proof of the following is straightforward.

THEOREM 4. T_n *is cosinusoidal over the integers.*

We now define for dyadic fractions $r=m/2^n$ functions C and S by equating $C(r)$ with a_m in T_n and $S(r)$ with b_m in T_n. We note that for any finite set of fractions, $C(r)$ and $S(r)$ are found as terms in T_n for some fixed n sufficiently large. We may easily prove

THEOREM 5. *Functions* $C(r)$ *and* $S(r)$ *are cosinusoidal over the set of dyadic fractions. Both have period* 1. *In their first quadrant,* $0\leqq r\leqq 1/4$, $C(r)$ *decreases from* 1 *to* 0 *and* $S(r)$ *increases from* 0 *to* 1.

For $|r|<1/8$, we have $C(r)>1/\sqrt{2}$ and $|S(r/2)|=|S(r)|/2C(r/2)<|S(r)|/\sqrt{2}$. Hence $|S(r/4)|<|S(r/2)|/\sqrt{2}<|S(r)|/2$, etc. We obtain

THEOREM 6. $\lim_{r\to 0} S(r)=0$ *and hence* $\lim_{r\to 0} C(r)=1$.

COROLLARY. $C(r)$ *and hence* $S(r)$ *are uniformly continuous over the dyadic fractions.*

Proof. We can make $|C(r_1)-C(r_2)|=2|S(r_1+r_2)S(r_1-r_2)|<2|S(r_1-r_2)|$ arbitrarily small by choosing $|r_1-r_2|$ sufficiently small.

The corollary permits us to extend functions C and S to the real domain. We define $c(x)=\lim_{r\to x} C(r)$ and $s(x)=\lim_{r\to x} S(r)$. We then have

THEOREM 7. *Functions $c(x)$ and $s(x)$ are cosinusoidal over the reals. They inherit the properties listed in Theorems* 5 *and* 6.

In proving the following theorem we must introduce a definition of π.

THEOREM 8. $\lim_{x\to 0} s(x)/x=2\pi$.

Proof. We may assume $x>0$. We consider the average $M_n=(1/n)\sum_{k=1}^{n} c(k/4n)$. In Quadrant I, $c(x)$ decreases and it is concave since $c(2x)+c(2y)=2c(x+y)c(x-y)<2c(x+y)$. It thus follows that M_n increases with n. Since $M_n<1$ for all n, $\lim_{n\to\infty} M_n$ exists. We define $\pi=2/\lim M_n$. We have by (12) that

$$nM_n = c\left(\frac{n+1}{8n}\right)s\left(\frac{1}{8}\right) \Big/ s\left(\frac{1}{8n}\right).$$

Hence $8ns(1/8n)=8c(n+1/8n)s(1/8)/M_n\to 4c(1/8)s(1/8)\pi=2\pi$ as $n\to\infty$.

We have shown, for $x_n=(1/8n)$, that $s(x_n)/x_n\to 2\pi$. But for $0<x<1/8$, we may choose n such that $x_{n+1}\leqq x<x_n$. Then $s(x_{n+1})/x_n<s(x)/x<s(x_n)/x_{n+1}$ and, since the outer fractions have a limit 2π, the theorem holds.

We now define $\cos x=c(x/2\pi)$ and $\sin x=s(x/2\pi)$. We have returned to traditional ground.

THEOREM 9. Cos x *and* sin x *are cosinusoidal over the reals. They are uniformly continuous. They have period 2π and are monotonic in the first quadrant,* $0\leqq x\leqq \pi/2$. *We have* $\lim_{x\to 0}(\sin x/x)=1$.

The initial restriction $a_1^2+b_1^2=1$ in (1) serves, of course, to eliminate FTD sequences associated with functions which are products of exponential and circular functions. The independent variables we have used may be identified with angles if we wish. We may let $P_0=(1, 0)$ and $P_n=(a_n, b_n)$ where $\{P_n\}$ is uniformly spaced on the unit circle and use elementary geometry to establish that $\{a_0, a_1, \cdots; b_0, b_1, \cdots\}$ is an FTD sequence.

Our introductory paragraph did not refer to the method of using functional equations. That such a treatment can be elegant is shown by the paper *A New Approach to Circular Functions, π, and* lim(sin x)/x by G. B. Robison in the March 1968 issue of this MAGAZINE. For the purpose of founding analysis, however, such methods have the undesirable feature of assuming that solutions of the functional equation(s) exist.

A NOTE ON LINEARITY

H. BURKILL, University of Sheffield

1. Dieudonné's book [1] on real and complex analysis has led to a widespread interest in the modern theory of differentiation, which is now included in many undergraduate curricula. The basis of this theory is the notion of linearity;

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1. Dieudonné's book [**1**] on real and complex analysis has led to a widespread interest in the modern theory of differentiation, which is now included in many undergraduate curricula. The basis of this theory is the notion of linearity;

and the purpose of the present note is to show how the teacher can illustrate the idea by using it to explain two elementary results on complex numbers and functions. The first is the isomorphism of the set $\boldsymbol{C}$ of complex numbers with the set of real 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}; \tag{1}$$

the second is the Cauchy-Riemann condition for the differentiability of a complex function.

2. Let ϕ be the natural bijection between the points of the Euclidean plane $\boldsymbol{R}^2$ and those of $\boldsymbol{C}$. If $E\subseteq \boldsymbol{R}^2$, we denote the corresponding subset of $\boldsymbol{C}$ by E^*. The bijection ϕ between points induces a bijection Φ between the functions with domain and range in $\boldsymbol{R}^2$ and those with domain and range in $\boldsymbol{C}$: a function f: $E\rightarrow \boldsymbol{R}^2$, where $E\subseteq \boldsymbol{R}^2$, is of the form $f=(u, v)^T$, where u, v are real-valued functions on E; the corresponding function f^*: $E^*\rightarrow \boldsymbol{C}$ is then $f^*=u^*+iv^*$, where the real-valued functions u^*, v^* on E^* are given by

$$u^*(z) = u(x, y), \qquad v^*(z) = v(x, y) \quad (z = x + iy \in E^*).$$

It is easy to see that

$$(af_1 + bf_2)^* = af_1^* + bf_2^* \quad (a, b \in \boldsymbol{R}^1), \qquad (f_1 \circ f_2)^* = f_1^* \circ f_2^*.$$

When $\boldsymbol{R}^2$ and $\boldsymbol{C}$ are also endowed with their usual metrics, then ϕ is an isometry between the resulting metric spaces. One consequence is that corresponding functions f, f^* are either both continuous or both discontinuous.

As *vector spaces* $\boldsymbol{R}^2$ and $\boldsymbol{C}$ differ: $\boldsymbol{R}^2$ is two-dimensional (over $\boldsymbol{R}^1$), and $\boldsymbol{C}$ is one-dimensional (over itself). It is therefore to be expected that Φ does not make linear functions correspond to linear functions. In fact, if the complex function $g^*=\lambda^*+i\mu^*$ is linear, then the corresponding function $g=(\lambda, \mu)^T$: $\boldsymbol{R}^2\rightarrow \boldsymbol{R}^2$ is linear, but when g is linear, then g^* is not necessarily linear. If λ, μ are given by

$$\lambda(x, y) = ax + by, \qquad \mu(x, y) = cx + dy \quad ((x, y)^T \in \boldsymbol{R}^2), \tag{2}$$

then a necessary and sufficient condition for g^* to be linear is that $g^*(iz)=ig^*(z)$ for all $z\in \boldsymbol{C}$, i.e., that

$$a = d \quad \text{and} \quad b = -c.$$

When this condition is satisfied, then

$$g^*(z) = \alpha z \qquad (z \in \boldsymbol{C}), \tag{3}$$

where $\alpha=a+ib$.

We denote the set of complex linear functions by $\mathfrak{L}^*$ and the set of corresponding linear functions on $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$ by $\mathfrak{L}$. This correspondence is an isomorphism with respect to addition and composition.

The linear function $g=(\lambda, \mu)^T$ given by (2) is represented by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The correspondence between the set of all linear functions on R^2 to R^2 (of which $\mathcal{L}$ is a proper subset) and the set of all 2×2 matrices is an isomorphism in which addition and composition of linear functions correspond to matrix addition and multiplication, respectively. It follows that both $\mathcal{L}$ and $\mathcal{L}^*$ are isomorphic with the set M of real matrices of the form (1). But $\mathcal{L}^*$ is also isomorphic with C (with addition and composition in $\mathcal{L}^*$ corresponding to addition and multiplication in C). Hence the bijection between C and M in which $a+ib$ corresponds to the matrix (1) is an isomorphism with respect to addition and multiplication in the two systems.

3. The function $f=(u, v)^T$: $E\to R^2(E\subseteq R^2)$ is differentiable at the interior point $(x_0, y_0)^T$ of E if there exists a linear function g: $R^2\to R^2$ such that

$$(4)\qquad \|f(x_0+\xi, y_0+\eta) - f(x_0, y_0) - g(\xi, \eta)\|/\|(\xi, \eta)\| \to 0 \quad \text{as} \quad \|(\xi, \eta)\| \to 0,$$

where $\|\cdot\|$ denotes the usual norm of R^2. When f is differentiable at $(x_0, y_0)^T$, then the partial derivatives of u, v exist at $(x_0, y_0)^T$ and g is represented by the matrix

$$\begin{pmatrix} D_1u(x_0, y_0) & D_2u(x_0, y_0) \\ D_1v(x_0, y_0) & D_2v(v_0, y_0) \end{pmatrix}.$$

Now (4) is equivalent to

$$\{f^*(z_0+\zeta) - f^*(z_0) - g^*(\zeta)\}/\zeta \to 0 \quad \text{as} \quad \zeta \to 0,$$

where $z_0=x_0+iy_0$, $\zeta=\xi+i\eta$; and, by the discussion of the previous paragraph, g^* is linear (i.e., of the form (3)) if and only if g is linear and also

$$(5)\qquad D_1u(x_0, y_0) = D_2v(x_0, y_0), \qquad D_2u(x_0, y_0) = -D_1v(x_0, y_0).$$

Furthermore, the differentiability of u and v is a necessary and sufficient condition for $f=(u, v)^T$ to be differentiable ([2], p. 101). Hence we arrive at the well-known proposition that the complex function $f^*=u^*+iv^*$ on E^* is differentiable at the interior point $z_0=x_0+iy_0$ of E^* if and only if the functions u, v: $E\to R^1$ are differentiable at $(x_0, y_0)^T$ and satisfy the Cauchy-Riemann equations (5).

References

1. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
2. W. H. Fleming, Functions of Several Variables, Addison-Wesley, Reading, 1965.

A REPRESENTATION FORMULA FOR THE SOLUTIONS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATION

E. J. PELLICCIARO, University of Delaware

Let p and q be continuous functions on a bounded closed interval I containing x_0. It is the purpose of this paper to give a simple existence and uniqueness proof for the initial value problem

$$(1)\qquad y'' - py' - qy = 0, \qquad y(x_0) = y_0, \qquad y'(x_0) = y_0',$$

The correspondence between the set of all linear functions on R^2 to R^2 (of which $\mathcal{L}$ is a proper subset) and the set of all 2×2 matrices is an isomorphism in which addition and composition of linear functions correspond to matrix addition and multiplication, respectively. It follows that both $\mathcal{L}$ and $\mathcal{L}^*$ are isomorphic with the set M of real matrices of the form (1). But $\mathcal{L}^*$ is also isomorphic with C (with addition and composition in $\mathcal{L}^*$ corresponding to addition and multiplication in C). Hence the bijection between C and M in which $a+ib$ corresponds to the matrix (1) is an isomorphism with respect to addition and multiplication in the two systems.

3. The function $f=(u, v)^T$: $E\rightarrow R^2(E\subseteq R^2)$ is differentiable at the interior point $(x_0, y_0)^T$ of E if there exists a linear function g: $R^2\rightarrow R^2$ such that

$$\|f(x_0+\xi, y_0+\eta)-f(x_0, y_0)-g(\xi, \eta)\|/\|(\xi, \eta)\| \rightarrow 0 \quad \text{as} \quad \|(\xi, \eta)\| \rightarrow 0, \tag{4}$$

where $\|\cdot\|$ denotes the usual norm of R^2. When f is differentiable at $(x_0, y_0)^T$, then the partial derivatives of u, v exist at $(x_0, y_0)^T$ and g is represented by the matrix

$$\begin{pmatrix} D_1u(x_0, y_0) & D_2u(x_0, y_0) \\ D_1v(x_0, y_0) & D_2v(v_0, y_0) \end{pmatrix}.$$

Now (4) is equivalent to

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Furthermore, the differentiability of u and v is a necessary and sufficient condition for $f=(u, v)^T$ to be differentiable ([2], p. 101). Hence we arrive at the well-known proposition that the complex function $f^*=u^*+iv^*$ on E^* is differentiable at the interior point $z_0=x_0+iy_0$ of E^* if and only if the functions u, v: $E\rightarrow R^1$ are differentiable at $(x_0, y_0)^T$ and satisfy the Cauchy-Riemann equations (5).

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$$y''-py'-qy=0, \qquad y(x_0)=y_0, \qquad y'(x_0)=y_0', \tag{1}$$

y_0 and y_0' arbitrary numbers. A product of the proof is a representation formula for the solution of (1). That the representation formula is indeed a solution under the hypothesis stated can be verified by direct substitution into (1).

A function ϕ on I is a solution of (1) only if ϕ' and ϕ'' exist on I and

$$\phi''(x) - p(x)\phi'(x) - q(x)\phi(x) = 0, \qquad x \in I.$$

From this, then,

$$D_x\phi'(x) \exp\left(-\int_{x_0}^{x} p(v)dv\right) = q(x)\phi(x) \exp\left(-\int_{x_0}^{x} p(v)dv\right),$$

so that, since ϕ is continuous on I,

$$\phi'(x) \exp\left(-\int_{x_0}^{x} p(v)dv\right) = y_0' + \int_{x_0}^{x} q(s)\phi(s) \exp\left(-\int_{x_0}^{s} p(v)dv\right) ds.$$

Thus, upon setting $E(x) = \exp \int_{x_0}^{x} p(v)dv$ and $E^{-1}(x) = 1/E(x)$, one has

(2) $$\phi'(x) = E(x)\left[y_0' + \int_{x_0}^{x} E^{-1}(s)q(s)\phi(s)ds\right],$$

which implies

(3) $$\phi(x) = y_0 + y_0' \int_{x_0}^{x} E(t)dt + \int_{x_0}^{x}\int_{x_0}^{t} E(t)E^{-1}(s)q(s)\phi(s)dsdt$$

for $x \in I$. Conversely now, suppose ϕ is continuous on I with (3) holding for $x \in I$. Then it follows from (3) that ϕ' exists and in fact that ϕ' is given on I by (2). Whence, from (2), ϕ'' exists and, by simple inspection, $\phi'' = p\phi' + q\phi$; moreover, from (3), $\phi(x_0) = y_0$, and from (2), $\phi'(x_0) = y_0'$. Thus, ϕ is a solution of (1), completing the proof of the following:

THEOREM 1. *A continuous function ϕ on I is a solution of* (1) *if and only if* (3) *holds for $x \in I$.*

Thus the problem of solving (1) is equivalently replaced by the problem of finding a function ϕ continuous on I for which (3) holds for $x \in I$. The form of (3), an integral equation, is considerably simplified with the introduction of the operator S defined for continuous functions f on I by

$$Sf(x) = \int_{x_0}^{x}\int_{x_0}^{t} E(t)E^{-1}(s)q(s)f(s)dsdt, \qquad x \in I.$$

Indeed, (3) is a special case of $\phi(x) = f(x) + S\phi(x)$, $x \in I$, or equivalently,

(4) $$\phi = f + S\phi,$$

where f is restricted to the set of continuous functions on I. The operator S, and therefore the corresponding integral in (3), plays a singularly important role in generating a function ϕ satisfying (4), hence in solving (1).

Note that Sf is a continuous function on I if f is, in which case so is SSf.

With this in mind, operators S^n, iterants of S, are defined inductively for continuous functions f on I in the following way. Set

$$S^0 f = f, \qquad S^1 f = Sf,$$

and then

$$S^n f = SS^{n-1} f, \qquad n = 1, 2, \cdots.$$

Pertinent properties of S^n are given below in the form of three lemmas. As their proofs are straightforward, they are omitted.

LEMMA 1. *S^n is a linear operator. That is, if $f_1, \cdots, f_k$ are continuous functions on I, then*

$$S^n(c_1 f_1 + \cdots + c_k f_k) = c_1 S^n f_1 + \cdots + c_k S^n f_k$$

for any set $c_1, \cdots, c_k$ of numbers.

LEMMA 2. *Let f be continuous on I, hence bounded on I by say M. Then $S^n f$ is continuous on I; moreover, there exists K such that $|S^n f(x)| \leq MK^n/n!$ for $n=0, 1, 2, \cdots$ and $x \in I$.*

LEMMA 3. *If f is continuous on I, then the series $\sum_{n=0}^{\infty} S^n f$ converges uniformly to a continuous function on I.*

Having disposed of the preliminaries, we give below the main theorem concerning (4):

THEOREM 2. *Let f be continuous on I. Then there exists one and only one continuous function ϕ on I such that $\phi = f + S\phi$;* indeed, $\phi = \sum_{n=0}^{\infty} S^n f$.

Proof. Define

$$\phi_0 = f$$

and then

$$\phi_n = f + S\phi_{n-1}, \qquad n = 1, 2, \cdots.$$

In view of Lemma 1,

$$\phi_n = \sum_{k=0}^{n} S^k f.$$

But the sequence $\{\phi_n\}$ is the sequence of partial sums defining the series $\sum_{n=0}^{\infty} S^n f$, which by Lemma 3 converges uniformly to a continuous function ϕ on I. That $\phi = \sum_{n=0}^{\infty} S^n f$ satisfies (4) follows from

$$\begin{aligned} f + S\phi &= f + S \sum_{n=0}^{\infty} S^n f \\ &= f + \sum_{n=0}^{\infty} S^{n+1} f \\ &= S^0 f + \sum_{n=1}^{\infty} S^n f = \phi, \end{aligned}$$

the second equality holding because of the uniform convergence on I of $\sum S^n f$. To show that it is the only such continuous function, suppose there are two, say ϕ and ψ. Then

$$\phi - \psi = S(\phi - \psi).$$

But this implies by induction that

$$\phi - \psi = S^n(\phi - \psi), n = 1, 2, \cdots.$$

This in turn implies by Lemma 2 that, for some M and K,

$$|\phi(x) - \psi(x)| \leqq \frac{MK^n}{n!}, \qquad n = 0, 1, 2, \cdots \quad \text{and} \quad x \in I.$$

Hence, for $x \in I$,

$$|\phi(x) - \psi(x)| \leqq \lim_{n\to\infty} \frac{MK^n}{n!} = 0,$$

implying $\phi = \psi$.

COROLLARY. *A function ϕ is a solution of* (1) *if and only if*

$$\phi(x) = \sum_{n=0}^{\infty} S^n\left(y_0 + y_0' \int_{x_0}^{x} E(r)dr\right), \qquad x \in I. \tag{5}$$

A pair of linearly independent solutions of (1) obtained from (5) is the pair

$$\begin{aligned} y_1(x) = 1 &+ \int_{x_0}^{x}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)ds_1dt_1 \\ &+ \int_{x_0}^{x}\int_{x_0}^{t_2} E(t_2)E^{-1}(s_2)q(s_2)\int_{x_0}^{s_2}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)ds_1dt_1ds_2dt_2 + \cdots, \end{aligned}$$

$$\begin{aligned} y_2(x) = \int_{x_0}^{x} E(r)dr &+ \int_{x_0}^{x}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)\int_{x_0}^{s_1} E(r)drds_1dt_1 \\ &+ \int_{x_0}^{x}\int_{x_0}^{t_2} E(t_2)E^{-1}(s_2)q(s_2)\int_{x_0}^{s_2}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1) \\ &\int_{x_0}^{s_1} E(r)drds_1dt_1ds_2dt_2 + \cdots. \end{aligned}$$

It is not at all difficult to compute ϕ' directly from (5), formally or by reference to convergence theorems, to obtain (2) and then to conclude as in the last part of the proof of Theorem 1 that ϕ as given by (5) is a solution of (1), by direct verification.

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A NOTE ON THE EQUIVALENCE OF FIVE THEOREMS IN ANALYSIS

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While all true theorems are logically equivalent (given the axioms) it is frequently useful to demonstrate particular paths by which a cluster of related theorems can be shown to imply one another cyclically. In this note we point out that it can be shown that the Weierstrass approximation theorem⇒mean-square convergence of Fourier series of continuous, periodic functions⇒uniform convergence of Fourier series of continuously differentiable periodic functions ⇒the uniform convergence on finite intervals of the Fourier integrals of integrable, continuously differentiable functions

$$\Rightarrow u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\alpha\xi}d\xi e^{i\alpha x-\alpha^2 t}d\alpha$$

is a solution to the heat equation $u_{xx}=u_t$ with initial condition

$$u(x, 0) = f(x)$$

⇒the Weierstrass approximation theorem, thereby completing the cycle.

The arguments used are relatively elementary, the most complicated result cited being the fact that continuous functions can be uniformly approximated by continuously differentiable ones which is used in the final implication.

Use of the Weierstrass theorem to show mean square convergence of Fourier series to continuous periodic functions is quite usual and can be found, for instance, in [1]. To show uniform convergence of Fourier series to continuously differentiable periodic functions (following an argument given in [1]), we proceed by computing Fourier coefficients γ_n for the continuous derivative, f', and integrating by parts to get $\gamma_n = inC_n$, where C_n are the Fourier coefficients of the function f. From mean-square completeness of the Fourier series we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\gamma_n|^2 = \sum_{n=-\infty}^{\infty} n^2 |C_n|^2.$$

Employing Schwartz's inequality we have

$$\begin{aligned}\sum_{0<p\leq|n|\leq q} |C_n e^{inx}| &= \sum_{0<p\leq|n|\leq q} \left|\frac{1}{in}\gamma_n e^{inx}\right| \\ &\leq \left(\sum_{0<p\leq|n|\leq q}\left(\frac{1}{n^2}\right)\right)^{1/2}\left(\sum_{0<p\leq|n|\leq q}(\gamma_n)^2\right)^{1/2} \\ &\leq \sqrt{\frac{\pi}{3}}\left(\int_{-\pi}^{\pi}|f'(x)|^2 dx\right)^{1/2}\end{aligned}$$

which establishes uniform convergence.

While "heuristic" arguments leading from Fourier series to Fourier integrals are frequently presented in textbooks, they are seldom carried through in a rigorous manner because the authors feel the results can be established more easily by proceeding directly. Zygmund, however, ([2], Vol. II, Chapter 16, Section 1, Theorem 1.3 et seq.) gives an elegant, simple and elementary proof that pointwise convergence and uniform convergence on finite intervals reduce to the same question for the Fourier series. This proof involves only elementary results, such as the mean-value theorems and the Riemann-Lebesgue lemma. Using Zygmund's approach we can easily establish that the Fourier integral of a continuously differentiable and absolutely integrable function converges uniformly to the functions on every finite interval.

Using this fact it is easily established that the function

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\alpha\xi}d\xi e^{i\alpha x-\alpha^2 t}d\alpha$$

satisfies the conditions

$$u_{xx} = u_t \qquad \text{for } t > 0$$

and

$$u(x, 0) = f(x)$$

whenever $f(x)$ is continuously differentiable and (absolutely) integrable.

It is also readily verified that the function $u(x, t)$ is analytic with respect to x and t, all x and for $t > 0$. Given a continuous function on a compact interval, it can be approximated uniformly by an integrable, continuously differentiable function f. By choosing sufficiently small t, this in turn is uniformly approximated by $u(x, t)$, and since u is entire analytic in x, it is uniformly approximated by polynomials in x, which establishes the Weierstrass theorem. The last two steps of our cycle of implications are based upon arguments given by F. John in [3].

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AN ALL-PURPOSE, 'FLOATING-POINT' CHART FOR THE ELEMENTARY ARITHMETIC OPERATIONS

C. R. WYLIE, JR., University of Utah

1. Introduction. In this day of slide rules, desk calculators, and electronic computers, there is little or no practical justification for a chart designed to expedite the elementary arithmetic operations. However, a chart consisting of rigorous manner because the authors feel the results can be established more easily by proceeding directly. Zygmund, however, ([2], Vol. II, Chapter 16, Section 1, Theorem 1.3 et seq.) gives an elegant, simple and elementary proof that pointwise convergence and uniform convergence on finite intervals reduce to the same question for the Fourier series. This proof involves only elementary results, such as the mean-value theorems and the Riemann-Lebesgue lemma. Using Zygmund's approach we can easily establish that the Fourier integral of a continuously differentiable and absolutely integrable function converges uniformly to the functions on every finite interval.

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1. Introduction. In this day of slide rules, desk calculators, and electronic computers, there is little or no practical justification for a chart designed to expedite the elementary arithmetic operations. However, a chart consisting of a single curved scale on which addition, subtraction, multiplication, division, and the extraction of square roots can all be carried out in a simple fashion with respect to an arbitrary point of the scale as origin and an arbitrary interval of the scale as unit interval would seem to be of some mathematical and pedagogical interest. Such a device is described in this note.

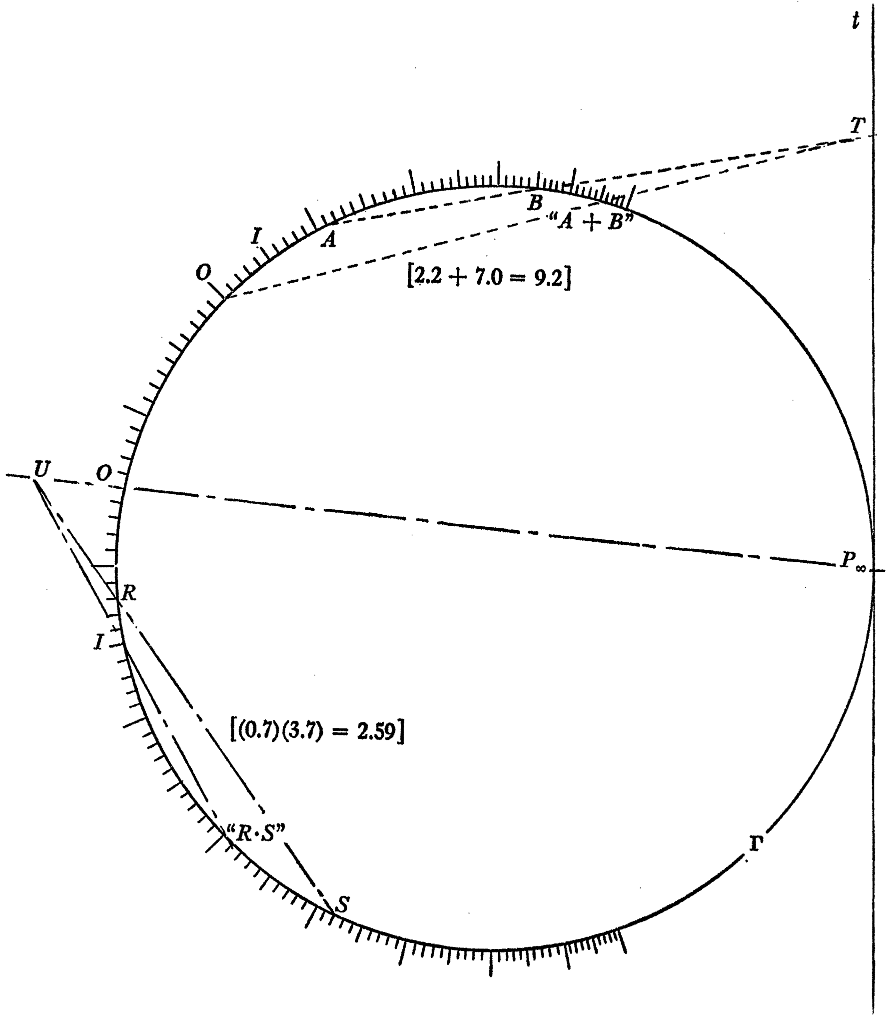

FIG. 1.

2. The construction and use of the chart. Let l be an arbitrary line bearing a uniform scale, let Γ be an arbitrary nonsingular conic, let P be an arbitrary point on Γ distinct from the intersections of l and Γ (if any), and let the scale on l be projected onto Γ from P. The chart then consists of Γ, as now graduated, together with the tangent t to Γ at the second intersection, P_∞, of Γ and the line through P which is parallel to l. Figure 1 shows a chart corresponding to the choices $l{:}x=0$, $\Gamma{:}x^2+y^2=1$, $P{:}(1, 0)$.

To use the chart to find the sum of two numbers, a and b, first select an arbitrary point of Γ, distinct from P_∞, to be the origin O of the scale on Γ, choose an arbitrary unit point I, also distinct from P_∞, and locate the points, A and B, which correspond to the values a and b. Next determine the intersection T of AB and the tangent t. The sum $a+b$ is then the scale value of the second point in which the line OT intersects Γ. This construction is illustrated in Figure 1.

The obvious way to compute $a-b$ is to find the sum of a and $-b$, as described above. Alternatively, we can locate the points A and B corresponding respectively to the numbers a and b and then solve the equation $x+b=a$ by first determining the intersection T of OA and t, and then reading $x=a-b$ from the second intersection of BT and Γ.

To use the chart to find the product of two numbers, r and s, first locate the corresponding points R and S with respect to an arbitrary origin O, and an arbitrary unit interval OI. Next determine the intersection U of RS and OP_∞. The product rs can then be read from the second intersection of UI and Γ. This construction is also illustrated in Figure 1.

The obvious way to compute the quotient r/s is to compute the product of r and $1/s$. Alternatively, we can solve the equation $sx=r$ by locating R and S, then determining the intersection of OP_∞ and RI, and finally reading $x=r/s$ from the second intersection of OS and Γ. It is interesting to note that the construction for r/s leads to P_∞ if $s=0$ and $r\neq 0$. If $r=s=0$, the construction is appropriately indeterminate.

To compute $\sqrt{r}$, that is, to solve the equation $x^2=r$, we first locate the point R corresponding to the number r, and then we determine the intersection U of RI and OP_∞. Finally, we read $\pm\sqrt{r}$ from the points of contact of the tangents to Γ from U. It is easy to verify that if r is negative, that is, if R is a point on the opposite side of OP_∞ from I, then U lies in the interior of Γ and no tangents can be drawn from it to Γ. Thus if $r<0$, $\sqrt{r}$ is 'imaginary.'

3. The mathematics behind the chart. Each of the properties of the chart we have described can be verified by at worst a tedious application of the processes of elementary analytic geometry. However, a more elegant and satisfying justification is provided by certain results from projective geometry.

It is well known that if three distinct points, P_0, P_1, P_ω are chosen arbitrarily on a line l, in the projective plane, and if the sum of two points, A and B, is defined to be the mate of P_0 in the involution in which A and B are corresponding points and P_ω is self-corresponding, and if the product of two points, R and S, is defined to be the mate of P_1 in the involution in which (R, S) and (P_0, P_ω) are two pairs of corresponding points, then under these operations the set of points belonging to $l-P_\omega$ is a field. Furthermore, if $l-P_\omega$ is a line of the euclidean plane, that is, if P_ω is the ideal point on l, then the points $P_0, P_1, P_2, P_3, \cdots$ define a uniform scale on $l-P_\omega$ which can be generated equally well by taking any one of its points to be P_0. When the scale on l is projected onto a conic Γ, from a point P on Γ, the point P_ω projects into the point P_∞ which is the second intersection of Γ and the line on P which is parallel to l, and of course the involutions which define sums and products on l become corresponding involutions on Γ. Finally, since an involution on a nonsingular conic is simply a perspective transformation of the conic into itself, the justification of our constructions is clear.

EVALUATION OF DOUBLE INTEGRALS BY MEANS OF THE DEFINITION

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Quite often when a beginning calculus student studies the double integral, he is confronted with statements such as the following: "The definition of the double integral is useless as a tool for evaluation in any particular case." [2, p. 743]. "As in functions of one variable, it is difficult or impossible to find the value of a double integral from its definition alone." [3, p. 630]. "Its usefulness as a working tool would be limited, however, if it were necessary to resort to the limit of sums to find numerical answers to specific problems." [4, p. 731].

How is a student to interpret statements such as these? Is he to disregard the definition of the double integral or is he to get the impression that it can never be used to evaluate a double integral? The examples presented in this paper should aid the beginning calculus student to understand the definition of the double integral, to see how it actually can be used to evaluate a double integral, and to see an application of the double integral.

DEFINITION. *Consider a function f of two variables defined over a region R in the xy-plane. With lines parallel to the coordinate axes, form rectangles that will cover R. Let n be the number of rectangles that lie wholly inside of R or on the boundary of R. These n rectangles are said to form a partition of R. For $i=1, 2, 3, \cdots, n$, let A_i be the area of the ith rectangle and (x_i, y_i) be an arbitrary point in this rectangle. Let δ, called the norm of the partition of R, be the length of the longest diagonal of the n rectangles of the partition of R. Let $V_n=\sum_{i=1}^{n} f(x_i, y_i)A_i$. If $\lim_{\delta\to 0} V_n$ exists, then f is said to be integrable over R and this limit is called the double integral of f over R, it being denoted by $\iint_R f(x, y)dR$.*

In work beyond the beginning calculus course, two important things are proved about the double integral. They are:

(1) If f is bounded and continuous over R, a region with area, then f is integrable over R [1, p. 338].

(2) The value of $\lim_{\delta\to 0} \sum_{i=1}^{n} f(x_i, y_i)A_i=\iint_R f(x, y)dR$ is independent of the partition and the arbitrary point chosen from each rectangle [1, p. 339].

These facts are used in the examples that follow, just as these facts are presented to the beginning calculus student when he studies the double integral.

Example 1.

$$\iint_R (x^2+y^2)dR = 2/3, \qquad R = \{(x, y) \mid 0 \leqq x \leqq 1 \text{ and } 0 \leqq y \leqq 1\}.$$

In the xy-plane, consider the lines $x=(j/p)$, $j=0, 1, 2, \cdots, p$ and $y=(k/p)$, $k=0, 1, 2, \cdots, p$ to form a partition of squares of R. The area of each square is $(1/p^2)$ and $\delta=\sqrt{2}/p$. Let $(j/p, k/p)$, $j=1, 2, 3, \cdots, p$ and $k=1, 2, 3, \cdots, p$ be the points in each square. Then,

$$V_n = \sum_{i=1}^{n} f(x_i, y_i)A_i,$$

$$= \sum_{k=1}^{p}\left(\sum_{j=1}^{p} f\left(\frac{j}{p}, \frac{k}{p}\right)\right)\frac{1}{p^2},$$

$$= \sum_{k=1}^{p}\left(\sum_{j=1}^{p}\left(\frac{j^2}{p^2} + \frac{k^2}{p^2}\right)\right)\frac{1}{p^2},$$

$$= \frac{1}{p^4}\sum_{k=1}^{p}\left(\sum_{j=1}^{p} j^2 + k^2\sum_{j=1}^{p} 1\right),$$

$$= \frac{1}{p^4}\sum_{k=1}^{p}\left(\frac{p(p+1)(2p+1)}{6} + k^2 p\right),$$

$$= \left(\frac{1}{p^4}\right)\frac{p(p+1)(2p+1)}{6}\sum_{k=1}^{p} 1 + \frac{1}{p^3}\sum_{k=1}^{p} k^2,$$

$$= \frac{p(p+1)(2p+1)}{6p^3} + \left(\frac{1}{p^3}\right)\frac{p(p+1)(2p+1)}{6},$$

$$V_n = \left(\frac{1}{3}\right)\left(1 + \frac{1}{p}\right)\left(2 + \frac{1}{p}\right).$$

When $p \to \infty$, then

$$\delta = \frac{\sqrt{2}}{p} \to 0.$$

Hence,

$$\iint_R (x^2 + y^2)\,dR = \lim_{p\to\infty}\left(\frac{1}{3}\right)\left(1 + \frac{1}{p}\right)\left(2 + \frac{1}{p}\right) = 2/3.$$

Example 2.

$$\iint_R (x^2 + y^2)\,dR = 1/6, \qquad R = \{(x, y) \mid 0 \leqq x \leqq 1 \text{ and } 0 \leqq y \leqq x - 1\}.$$

In the xy-plane, consider the lines $x = (j/p)$, $j = 0, 1, 2, \cdots, p$ and $y = (k/p)$, $k = 0, 1, 2, \cdots, p$ to form a partition of squares of R. The area of each square is $(1/p^2)$ and $\delta = (\sqrt{2}/p)$. Let $(j/p, k-1/p)$, $j = 0, 1, 2, \cdots, p-k$ and $k = 1, 2, 3, \cdots, p$ be the points in each square. Then,

$$V_n = \sum_{i=1}^{n} f(x_i, y_i) A_i,$$

$$= \sum_{k=1}^{p}\left(\sum_{j=0}^{p-k} f\left(\frac{j}{p}, \frac{k-1}{p}\right)\right)\frac{1}{p^2},$$

$$= \sum_{k=1}^{p}\left(\sum_{j=0}^{p-k}\left(\frac{j^2}{p^2} + \frac{(k-1)^2}{p^2}\right)\right)\frac{1}{p^2},$$

$$= \frac{1}{p^4}\sum_{k=1}^{p}\left(\sum_{j=0}^{p-k} j^2 + (k-1)^2\sum_{j=0}^{p-k} 1\right),$$

$$= \frac{1}{p^4}\sum_{k=1}^{p}\left(\frac{(p-k)(p-k+1)(2(p-k)+1)}{6} + (k-1)^2(p-k)\right),$$

$$= \left(\frac{-8}{6p^4}\right)\sum_{k=1}^{p} k^3 + \left(\frac{12p+15}{6p^4}\right)\sum_{k=1}^{p} k^2 - \left(\frac{6p^2+18p+7}{6p^4}\right)\sum_{k=1}^{p} k$$

$$+ \left(\frac{2p^3+3p^2+7p}{6p^4}\right)\sum_{k=1}^{p} 1,$$

$$V_n = \left(\frac{-1}{3}\right)\left(1+\frac{1}{p}\right)^2 + \left(\frac{1}{36}\right)\left(12+\frac{15}{p}\right)\left(1+\frac{1}{p}\right)\left(2+\frac{1}{p}\right)$$

$$- \left(\frac{1}{12}\right)\left(6+\frac{18}{p}+\frac{7}{p^2}\right)\left(1+\frac{1}{p}\right) + \left(\frac{1}{6}\right)\left(2+\frac{3}{p}+\frac{7}{p^2}\right).$$

When $p\to\infty$, then

$$\delta = \frac{\sqrt{2}}{p} \to 0.$$

Hence,

$$\iint_R (x^2+y^2)dR = \frac{-1}{3} + \frac{2}{3} - \frac{1}{2} + \frac{1}{3} = 1/6.$$

Example 3.

$$\iint_R (x+y)dR = 7/20, \quad R = \{x, y) \mid 0 \leqq x \leqq 1 \text{ and } 0 \leqq y \leqq x^2\}.$$

In the xy-plane, consider the lines $x=(j/p)$, $j=0, 1, 2, \cdots, p$ and $y=(k/p^2)$, $k=0, 1, 2, \cdots, p^2-1, p^2$ to form a partition of rectangles of R. The area of each rectangle is $1/p^3$ and $\delta=(1/p^2)\sqrt{p^2+1}$. Let $(p-k/p, j/p^2)$, $j=1, 2, 3, \cdots, (p-k)^2$ and $k=0, 1, 2, \cdots, p-1$ be the points in each rectangle. Then,

$$V_n = \sum_{i=1}^{n} f(x_i, y_i)A_i,$$

$$= \sum_{k=0}^{p-1}\left(\sum_{j=1}^{(p-k)^2} f\left(\frac{p-k}{p}, \frac{j}{p^2}\right)\right)\frac{1}{p^3},$$

$$= \sum_{k=0}^{p-1}\left(\sum_{j=1}^{(p-k)^2}\left(\frac{p-k}{p} + \frac{j}{p^2}\right)\right)\frac{1}{p^3},$$

$$= \frac{1}{p^3}\sum_{k=0}^{p-1}\left(\frac{p-k}{p}\sum_{j=1}^{(p-k)^2} 1 + \frac{1}{p^2}\sum_{j=1}^{(p-k)^2} j\right),$$

$$= \frac{1}{p^3}\sum_{k=0}^{p-1}\left(\left(\frac{p-k}{p}\right)(p-k)^2 + \frac{1}{p^2}\,\frac{(p-k)^2((p-k)^2+1)}{2}\right),$$

$$= \left(\frac{1}{2p^5}\right)\sum_{k=0}^{p-1} k^4 - \left(\frac{6p}{2p^5}\right)\sum_{k=0}^{p-1} k^3 + \left(\frac{12p^2+1}{2p^5}\right)\sum_{k=0}^{p-1} k^2$$
$$- \left(\frac{10p^3+2p}{2p^5}\right)\sum_{k=0}^{p-1} k + \left(\frac{3p^4+p^2}{2p^5}\right)\sum_{k=0}^{p-1} 1,$$

$$V_n = \left(\frac{1}{60}\right)\left(1-\frac{1}{p}\right)\left(2-\frac{1}{p}\right)\left(3\left(1-\frac{1}{p}\right)^2 + 3\left(\frac{1}{p}-\frac{1}{p^2}\right)-\frac{1}{p^2}\right)$$
$$- \left(\frac{3}{4}\right)\left(1-\frac{1}{p}\right)^2 + \left(\frac{1}{12}\right)\left(12+\frac{1}{p^2}\right)\left(1-\frac{1}{p}\right)\left(2-\frac{1}{p}\right)$$
$$- \left(\frac{1}{4}\right)\left(10+\frac{2}{p^2}\right)\left(1-\frac{1}{p}\right) + \left(\frac{3}{2}+\frac{1}{2p^2}\right)\left(1-\frac{1}{p}\right).$$

When $p\to\infty$, then $\delta=(1/p^2)\sqrt{p^2+1}\to 0$. Hence,

$$\iint_R (x+y)dR = \frac{1}{10}-\frac{3}{4}+2-\frac{5}{2}+\frac{3}{2} = 7/20.$$

In each of the above examples, the function f was chosen so that $f(x, y)\geqq 0$ for all (x, y) in R. In each example, V_n gives an approximation of the volume of the solid whose base is R and whose upper surface is the graph of f. Then $\lim_{\delta\to 0} V_n=\iint_R f(x, y)dR$ gives the exact volume. Hence the value of the double integral in each example is the volume of a solid. A student's intuition usually tells him that a plane, or any portion of a plane, has a volume $=0$. This can be readily shown by a double integral, evaluating it of course by means of the definition.

Example 4. Show that the portion of the plane $y=x$, bounded by the surface $z=x^2+y^2$, the plane $z=0$ and the plane $x+y=2$ has zero volume.

In terms of a double integral, this would be evaluating $\iint_R (x^2+y^2)dR$, where $R=\{(x, y)\mid 0\leqq x\leqq 1 \text{ and } y=x\}$. In the xy-plane, consider the lines $x=(i/n)$ and $y=(i/n)$, $i=0, 1, 2, \cdots, n$ to form a partition of the unit square in the first quadrant. Consider only those n squares which cover the region R. The area of each square is $(1/n^2)$ and $\delta=(\sqrt{2}/n)$. Let $(x_i, y_i)=(i/n, i/n)$, $i=1, 2, 3, \cdots, n$ be an arbitrary point in each square. Then,

$$V_n = \sum_{i=1}^{n} f(x_i, y_i)A_i,$$
$$= \sum_{i=1}^{n} f\left(\frac{i}{n}, \frac{i}{n}\right)\frac{1}{n^2},$$
$$= \sum_{i=1}^{n}\left(\frac{i^2}{n^2}+\frac{i^2}{n^2}\right)\frac{1}{n^2},$$
$$= \frac{2}{n^4}\sum_{i=1}^{n} i^2,$$
$$V_n = \left(\frac{1}{3}\right)\left(\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

When $n \to \infty$, then

$$\delta = \frac{\sqrt{2}}{n} \to 0.$$

Hence,

$$\iint_R (x^2 + y^2)dR = \left(\frac{1}{3}\right)(0)(1)(2) = 0.$$

Although the computations involved in each of these examples can be time-consuming, they certainly are not impossible to handle. These examples should help one to better understand the definition of the double integral, to show how it can be used to evaluate such, and to show one of its applications.

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AN ARITHMETICAL PROBLEM INVOLVING THE SUM OF INTEGERS IN A.P.

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1. Introduction. In [2], Sutcliffe considered the problem concerning integers $\overline{N}$ to base n obtained by reversing the digits of N such that $k\overline{N} = N$ for some integer k. Using the symbol $(a_i, a_{i-1}, \cdots, a_2, a_1)_n$, $a_i \neq 0$, to denote an integer of i digits to base n, the author obtained the following theorem for the 2-digit case:

"*If* $k(a, b)_n = (b, a)_n$, *then* $k(a, a+b, \cdots, a+b, b)_n = (b, a+b, \cdots, a+b, a)_n$, *where any number of digits equal to* $a+b$ *may be introduced between* a *and* b."

If we restrict our attention to integers to base 10, the problem is actually not new; in fact, for $k=4$ and 9, the solutions $4 \times 2178 = 8712$ and $9 \times 1089 = 9801$ were given by Burg [1, p. 464]. Somewhat similar to Sutcliffe's 2-digit case, it is interesting to note that if we insert the digit 9 any number of times between the first two digits and the last two digits in either of the above two solutions, the resulting integers again have the required property, that is,

$$4 \times 21(9_s)78 = 87(9_s)12, \qquad 9 \times 10(9_s)89 = 98(9_s)01,$$

where the symbol (9_s) denotes $\underbrace{99 \cdots 9}_{s \text{ times}}$, $s \geqq 1$.

When $n \to \infty$, then

$$\delta = \frac{\sqrt{2}}{n} \to 0.$$

Hence,

$$\iint_R (x^2 + y^2) dR = \left(\frac{1}{3}\right)(0)(1)(2) = 0.$$

Although the computations involved in each of these examples can be time-consuming, they certainly are not impossible to handle. These examples should help one to better understand the definition of the double integral, to show how it can be used to evaluate such, and to show one of its applications.

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AN ARITHMETICAL PROBLEM INVOLVING THE SUM OF INTEGERS IN A.P.

SAMUEL S. H. YOUNG, University of Hong Kong

1. Introduction. In [2], Sutcliffe considered the problem concerning integers $\overline{N}$ to base n obtained by reversing the digits of N such that $k\overline{N} = N$ for some integer k. Using the symbol $(a_i, a_{i-1}, \cdots, a_2, a_1)_n$, $a_i \neq 0$, to denote an integer of i digits to base n, the author obtained the following theorem for the 2-digit case:

"*If* $k(a, b)_n = (b, a)_n$, *then* $k(a, a+b, \cdots, a+b, b)_n = (b, a+b, \cdots, a+b, a)_n$, *where any number of digits equal to* $a+b$ *may be introduced between* a *and* b."

If we restrict our attention to integers to base 10, the problem is actually not new; in fact, for $k=4$ and 9, the solutions $4 \times 2178 = 8712$ and $9 \times 1089 = 9801$ were given by Burg [1, p. 464]. Somewhat similar to Sutcliffe's 2-digit case, it is interesting to note that if we insert the digit 9 any number of times between the first two digits and the last two digits in either of the above two solutions, the resulting integers again have the required property, that is,

$$4 \times 21(9_s)78 = 87(9_s)12, \qquad 9 \times 10(9_s)89 = 98(9_s)01,$$

where the symbol (9_s) denotes $\underbrace{99 \cdots 9}_{s \text{ times}}$, $s \geqq 1$.

We observe that there are other arithmetical problems, besides the ones mentioned above, which possess extensible solutions. This means that corresponding to a certain solution $N=LM$ of the problem, where LM denotes the integer formed by annexing the digits of M to the right of L, there are also solutions of the form $N_s=L(q_s)M$ for some digit q and any positive integer s. We say that $L(q_s)M$, $s\geqq 1$, are extended solutions of LM. In particular, $N(q_s)$, $s\geqq 1$, are extended solutions of N. (Note: The product of L by M will be denoted by $L\cdot M$.)

We shall treat in this paper an interesting arithmetical problem which possesses extensible solutions. It involves the sum of integers in A.P.

2. Statement of the problem. Consider a sequence of positive integers to base 10 in A.P. Let L be the first term, M the last term, d the common difference and N the sum of the sequence. Then

$$N = (L + M)\cdot(M - L + d)/2\cdot d. \tag{1}$$

If we require that the sum of the sequence is equal to the integer formed by annexing the digits of the last term to the right of the first term, then the problem possesses extensible solutions in pairs. In more detail, we assert that if $L=L'L''$, $M=M'M''$, then corresponding to a certain solution $N=LM$ of the problem, there are also solutions $N_s=L_sM_s$, where $L_s=L'(q_s)L''$, $M_s=M'(r_s)M''$, $s\geqq 1$, and the digits q and r may or may not be distinct.

3. General scheme of solution. Assume that M is an integer of m digits. Then $N=10^m\cdot L+M$, and by simple computation, we can put (1) in the following form:

$$(L + M + (10^m - 1)\cdot d)\cdot(L - M + 10^m\cdot d) = 10^m\cdot(10^m - 1)\cdot d^2. \tag{2}$$

If we denote the first and second factors on the left hand side of (2) by A and B respectively, we obtain

$$\begin{cases} L = (A + B - (2\cdot 10^m - 1)\cdot d)/2, \\ M = (A - B + d)/2. \end{cases} \tag{3}$$

Since $A-B=2\cdot M-d<2\cdot 10^m-d$, $A\cdot B=10^m\cdot(10^m-1)\cdot d^2$, we have

$$A^2 - (2\cdot 10^m - d)\cdot A - 10^m\cdot(10^m - 1)\cdot d^2 < 0. \tag{4}$$

The discriminant of the above quadratic expression in A is less than $(2\cdot 10^m-1)^2\cdot(d^2+1)$, and so $10^m\cdot(1+\sqrt{d^2+1})$ is an upper bound for A. Also, $A=10^m\cdot d+L+M-d>10^m\cdot d$. Thus,

$$10^m\cdot d < A < 10^m\cdot(1 + \sqrt{d^2 + 1}). \tag{5}$$

By (3), it is clear that A and B are both even if d is even but are of different parity if d is odd. With this observation and using (5) as a necessary condition, we can determine L and M so that $N=LM$ is a solution of our problem.

4. Some special cases. We give examples of some special cases as follows:

(i) $d=1$, $m=1$: For this simple case, the following solutions are well known and can be easily obtained without resorting to our method:

$$\text{a.}\quad 1+2+3+4+5=15,$$
$$\text{b.}\quad 2+3+4+5+6+7=27.$$

(ii) $d=1$, $m=2$: In this case, we have $A\cdot B=9900=2^2\cdot 3^2\cdot 5^2\cdot 11$. Since d is odd, either A or B must be odd. Furthermore, we have $100<A<242$ by (5). There are only five pairs of values (A, B) for which these conditions are satisfied, namely, (75, 132), (60, 165), (55, 180), (45, 220) and (44, 225). Correspondingly, we have indeed the following solutions to our problem:

$$\text{a.}\quad 4+5+\cdots+29=429,$$
$$\text{b.}\quad 13+14+\cdots+53=1353,$$
$$\text{c.}\quad 18+19+\cdots+63=1863,$$
$$\text{d.}\quad 33+34+\cdots+88=3388,$$
$$\text{e.}\quad 35+36+\cdots+91=3591.$$

(iii) $d=2$, $m=2$: Proceeding as in (ii), we obtain three solutions in the present case:

$$\text{a.}\quad 1+3+\cdots+21=121,$$
$$\text{b.}\quad 8+10+\cdots+58=858,$$
$$\text{c.}\quad 17+19+\cdots+85=1785.$$

For other values of d (positive or negative) and m, examples can be obtained by using a similar method.

5. Extensible solutions. Let us consider example 4(i)a. By annexing the digit 3 to the right of the first term and also to the right of the last term, the sum of the new sequence of consecutive integers is 1353, which is just 4(ii)b. In fact, 4(i)a has the following extended solutions:

$$1(3_s)+1(3_{s-1})4+\cdots+5(3_s)=1(3_s)5(3_s),\qquad s\geqq 1.$$

Not every solution of our problem is extensible, for instance 4(i)b. We shall show that all the three examples under 4(iii) are extensible, leaving the remaining examples under 4(ii) and other cases for the investigation of the reader.

For $d=2$, we have $A\cdot B=2^{m+2}\cdot 3^2\cdot 5^m\cdot(1_m)$. By (5), the value of A must lie between $2\cdot 10^m$ and $(1+\sqrt{5})\cdot 10^m$, and hence, $2^{m+1}\cdot 5^m<A<2^m\cdot 3^4\cdot 5^{m-2}$. Since A, B must both be even, the possible values of (A, B), without considering the factors of (1_m), are $(2^2\cdot 5\cdot(1_m),\ 2^m\cdot 3^2\cdot 5^{m-1})$, $(2^3\cdot 3\cdot(1_m),\ 2^{m-1}\cdot 3\cdot 5^m)$, $(2^m\cdot 3\cdot 5^m,\ 2^2\cdot 3\cdot(1_m))$ and $(2^{m-2}\cdot 3^2\cdot 5^m,\ 2^4\cdot(1_m))$. Corresponding to the first three pairs of factors, we have

$$\text{a.}\quad L=1(1_{m-2}),\qquad M=21(1_{m-2}),$$
$$\text{b.}\quad L=8(3_{m-2}),\qquad M=58(3_{m-2}),$$
$$\text{c.}\quad L=1(6_{m-2})7,\qquad M=8(3_{m-2})5,$$

which are extended solutions for examples a, b and c under 4(iii) respectively for $m \geqq 3$. The pair of factors $2^{m-2} \cdot 3^2 \cdot 5^m$ and $2^4 \cdot (1_m)$ leads to extended solutions of the following sequence:

$$139 + 141 + \cdots + 2363 = 1392363.$$

6. Concluding remarks. Our problem as well as that considered by Burg and Sutcliffe belong to a class of problems involving the properties of the digits of integers. Such problems are fascinating but are usually not very easy due to lack of general method of approach. By limiting the range of values to be considered, as illustrated in our discussion, solutions by elementary means can be achieved in some instances. Nevertheless, it is to be noted that there are seemingly simple problems of similar nature, solutions of which have resisted the efforts of even great mathematicians!

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A COMPARISON OF THE ARCHIMEDEAN AND COMPLETENESS PROPERTIES

JACK M. ROBERTSON, Washington State University

Another interesting equivalent statement of completeness for ordered fields was recently added by Moss and Roberts [1] to the already long list. The Archimedean property can also be stated in a similar way and an interesting comparison of the two properties results.

Let F be an ordered field.

DEFINITION 1. *A relation ρ is locally universal if for each $x \in F$ there is a neighborhood N_x of x such that $u\rho v$ whenever $u, v \in N_x$.*

DEFINITION 2. *A relation ρ is uniformly locally universal if there is an $a > 0$ such that for every $x \in F$, $u\rho v$ whenever $u, v \in (x-a, x+a)$.*

Note that a relation satisfying the second condition satisfies the first.

THEOREM 1. *F is complete if and only if every transitive locally universal relation is universal.*

Proof. See [1].

THEOREM 2. *F is Archimedean if and only if every transitive uniformly locally universal relation is universal.*

Proof. Suppose F is Archimedean and ρ is such a relation. Assume $x, y \in F$ with $x < y$. (If $x = y$ clearly $x\rho y$.) For some least integer n, $x + na \geqq y$ and using which are extended solutions for examples a, b and c under 4(iii) respectively for $m \geqq 3$. The pair of factors $2^{m-2} \cdot 3^2 \cdot 5^m$ and $2^4 \cdot (1_m)$ leads to extended solutions of the following sequence:

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Proof. Suppose F is Archimedean and ρ is such a relation. Assume $x, y \in F$ with $x < y$. (If $x = y$ clearly $x\rho y$.) For some least integer n, $x + na \geqq y$ and using the uniform local universal property of ρ it follows that

$$x\rho\left(x+\frac{a}{2}\right),$$

$$\left(x+\frac{a}{2}\right)\rho\left(x+\frac{3a}{2}\right),\ \cdots,\ \left(x+\left(n-\frac{3}{2}\right)a\right)\rho\left(x+\left(n-\frac{1}{2}\right)a\right),$$

$$\left(x+\left(n-\frac{1}{2}\right)a\right)\rho y.$$

So by transitivity $x\rho y$.

Conversely if F is not Archimedean, for some $a>0$ and b, $na<b$ for all natural numbers n. Define ρ as follows: $u\rho v$ if and only if either

(a) for some natural number m, $u<ma$ and $v<ma$, or

(b) for all natural numbers m, $u>ma$ and $v>ma$.

Then ρ is transitive and uniformly locally universal but a and b are not related.

Thus completeness requires that all locally universal transitive relations be universal regardless of the size of the neighborhoods, whereas the Archimedean property only requires universality in the case of uniform sized neighborhood universality.

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SEMINATURAL SYSTEMS AS NONEMPTY, WELL-ORDERED SETS WITHOUT LAST ELEMENT

SAMUEL T. STERN, State University College at Buffalo

R. A. Jacobson in [2] gives a set of three axioms which he shows to be equivalent to the axioms of a seminatural system as defined by this author in [1], and which he shows to be completely independent.

As stated at the beginning of [1], a seminatural system may be thought of as a nonempty, well-ordered set [3] without last element. Indeed, we may take as an equivalent set of axioms the following.

DEFINITION. *A set N, together with a binary relation R on N, is said to be a seminatural system if and only if the following axioms are satisfied:*

B-1 *N is a nonempty, well-ordered set with respect to R.*

B-2 *For every $x\in N$ there exists $y\in N$, $y\neq x$, such that $x\ R\ y$.*

It is easy to show that this pair of equally "intuitively appealing" axioms forms a completely independent set which is equivalent to the axiom set of [1]. For {B-1, B-2} we may take the set of natural numbers; for {∼B-1, B-2} the open interval of reals between 0 and 1; for {B-1, ∼B-2} the finite set {1, 2}; and for {∼B-1, ∼B-2} the closed interval of reals between 0 and 1; all examples in the uniform local universal property of ρ it follows that

$$x\rho\left(x+\frac{a}{2}\right),$$

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Proof of Theorem P-2. Let a be the first element of the nonempty subset N of N.

Proof of Theorem P-4. Let G be a nonempty subset of N such that for every $x \in N$, $I(x) \subset G$ implies $x \in G$. Suppose $G \neq N$. Then $N - G$ is nonempty and, by B-1, has a first element b. Then $I(b) \subset G$ and by hypothesis $b \in G$, a contradiction.

Proof of Theorem B-1. By P-2, N is nonempty. Let W be a nonempty subset of N and suppose that W has no first element. Then for each $x \in W$, there exists $y \in W$ such that yRx. Now $N - W$ is not empty for otherwise $W = N$ and W would have a first element by P-2. Let $x \in N$ such that $I(x) \subset N - W$. Then $x \in N - W$, for if $x \in W$, then there exists some $y \in W$ such that yRx and hence $y \in N - W$. By P-4, $N - W = N$ and W is empty, a contradiction. Hence W has a first element.

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A SHORT PROOF OF CRAMER'S RULE

STEPHEN M. ROBINSON, United States Army, University of Wisconsin, Madison

Many texts on linear algebra (e.g., [1] through [7]) prove Cramer's rule by using the relationship $A^{-1} = \text{adj } A/\det A$ and comparing cofactor expansions. The following proof may provide more insight into what is actually happening when Cramer's rule is used.

Let $Ax = b$, with A $n \times n$ and nonsingular. Let the columns of A be $a_1, \cdots, a_n$ and those of the identity be $e_1, \cdots, e_n$. Define X_k by

$$X_k = [e_1, \cdots, e_{k-1}, x, e_{k+1}, \cdots, e_n].$$

Then

$$x_k = \det X_k = \det A^{-1}AX_k = \det AX_k/\det A$$
$$= \det [a_1, \cdots, a_{k-1}, b, a_{k+1}, \cdots, a_n]/\det A,$$

which is Cramer's rule.

This proof makes it easier to see what we are doing when we use Cramer's rule. We want to evaluate $\det X_k$ in order to find x_k. But X_k contains the unobservable vector x. We therefore take the determinant of the image of X_k under the transformation represented by A, and then, to compensate for the transformation, divide the result by $\det A$.

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ON AN INTERESTING METRIC SPACE

R. SHANTARAM, SUNY at Stony Brook

In [1] Hildebrand and Milnes have defined on the euclidean plane the metric:

$$(1)\quad \rho_2(x, y) = \begin{cases} 0 & \text{if } x_1 = y_1 \text{ and } x_2 = y_2 \\ \frac{1}{2} & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \text{ or } x_1 \neq y_1 \text{ and } x_2 = y_2 \\ 1 & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2 \end{cases}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are members of R^2. The purpose of this note is twofold. First, to generalize this metric on R^2 and extend the definition to R^n; this is done in Sections 1 and 2. Second, to introduce a metric which is rotation invariant but not translation invariant. This is done in Section 3.

1. Let a and b be positive real numbers such that $b \leqq 2a \leqq 2b$. Define ρ_2^*: $R^2 \times R^2 \to R$ as $\rho_2^*(x, y) = 0$, a, or b respectively according as exactly both, exactly one or exactly none of the coordinate of x and y are equal. Then the following result is easily proved.

THEOREM 1. *For each fixed value of a and b as specified above ρ_2^* is a metric on R^2.*

We remark that for $a = \frac{1}{2}$ and $b = 1$ we obtain the metric in (1). The metric ρ_2^* gives rise to the following interesting neighborhoods of a point in R^2. An ϵ-neighborhood of $x \in R^2$ is the singleton set $\{x\}$ if $0 < \epsilon \leqq a$, the pair of straight lines thru x parallel to the coordinate axes if $a < \epsilon \leqq b$ and the whole plane if $\epsilon > b$.

2. We now extend the definition of ρ_2 to the n-dimensional case.

THEOREM 2. *Let $x, y \in R^n$, $n \geqq 2$. Define $\rho_n(x, y) = 1 - (k/n)$ if exactly k of the coordinates of x are equal to the corresponding coordinates of y ($k = 0, 1, \cdots, n$). Then ρ_n is a metric on R^n.*

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In [1] Hildebrand and Milnes have defined on the euclidean plane the metric:

$$(1)\quad \rho_2(x, y) = \begin{cases} 0 & \text{if } x_1 = y_1 \text{ and } x_2 = y_2 \\ \frac{1}{2} & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \text{ or } x_1 \neq y_1 \text{ and } x_2 = y_2 \\ 1 & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2 \end{cases}$$

where $x=(x_1, x_2)$ and $y=(y_1, y_2)$ are members of R^2. The purpose of this note is twofold. First, to generalize this metric on R^2 and extend the definition to R^n; this is done in Sections 1 and 2. Second, to introduce a metric which is rotation invariant but not translation invariant. This is done in Section 3.

1. Let a and b be positive real numbers such that $b \leqq 2a \leqq 2b$. Define ρ_2^*: $R^2 \times R^2 \to R$ as $\rho_2^*(x, y) = 0$, a, or b respectively according as exactly both, exactly one or exactly none of the coordinate of x and y are equal. Then the following result is easily proved.

THEOREM 1. *For each fixed value of a and b as specified above ρ_2^* is a metric on R^2.*

We remark that for $a=\frac{1}{2}$ and $b=1$ we obtain the metric in (1). The metric ρ_2^* gives rise to the following interesting neighborhoods of a point in R^2. An ϵ-neighborhood of $x \in R^2$ is the singleton set $\{x\}$ if $0<\epsilon \leqq a$, the pair of straight lines thru x parallel to the coordinate axes if $a<\epsilon \leqq b$ and the whole plane if $\epsilon > b$.

2. We now extend the definition of ρ_2 to the n-dimensional case.

THEOREM 2. *Let x, $y \in R^n$, $n \geqq 2$. Define $\rho_n(x, y) = 1-(k/n)$ if exactly k of the coordinates of x are equal to the corresponding coordinates of y ($k=0, 1, \cdots, n$). Then ρ_n is a metric on R^n.*

Proof. It is clear that $\rho_n(x, y) \geqq 0$ where equality holds iff $x = y$. Further ρ_n is symmetric. We need to prove the triangle inequality $\rho_n(x, z) \leqq \rho_n(x, y) + \rho_n(y, z)$. Put $D(x, y) = \{i | x_i \neq y_i\}$ and denote by $\#(x, y)$ the cardinality of $D(x, y)$. Then we have to prove that

$$\#(x, z) \leqq \#(x, y) + \#(y, z). \tag{2}$$

This follows by taking the cardinality of both sides of the relation $D(x, z) \subset D(x, y) \cup D(y, z)$ which is an easy consequence of the definition of D. The theorem is proved.

Remarks (i) For $n = 2$ in the theorem we get the metric in (1) (ii) For each n, ρ_n is translation invariant but not rotation invariant. (iii) The content (or volume) of a rectangular box in R^n whose edges are parallel to the coordinate axes is $1/n^n$ whereas that of a "tilted" one is k^k/n^n where k is the number of edges (emanating from any one vertex of the box) which are not parallel to any of the coordinate axes. (If $k = 0$, 0^0 is taken to be 1.) As a corollary we see that the content of a box none of whose edges is parallel to any of the coordinate axes is unity. (iv) According to this metric the content of a sub-box of a rectangular box may exceed the content of the box itself. (v) An ϵ-neighborhood of $x \in R^n$ is the singleton set containing x if $0 < \epsilon \leqq 1/n$. In general, for $k = 1, 2, \cdots, n-1$ if $k/n < \epsilon \leqq (k+1)/n$ it is the union of n hyperplanes (each of dimension k) thru the point x and parallel to one of the coordinate axes. It is the whole space if $\epsilon > 1$.

The generalization of ρ_n to ρ_n^* on the lines of Theorem 1 can be carried out. We merely state the result.

THEOREM 3. *Let* $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$ *be real numbers such that* $a_j + a_{n-j} \geqq a_n$ *for all* $j = 1, 2, \cdots, n$. *Then* $\rho_n^*(x, y) = a_k$ *where* $k = \#(x, y)$ *is a metric on* R^n.

If $a_k = k/n$ we get the metric ρ_n. An example of a set of values for a_k (different from k/n) having the properties in the theorem is $a_k = k + 1 - \lambda$, $(0 \leqq \lambda \leqq 1)$. If $n = 2$, $a_1 = a$, $a_2 = b$ then we get ρ_2^*.

3. The metrics considered earlier are invariant under translation of the axes parallel to themselves but are not rotation invariant. In this section we define a metric which is rotation, but not translation, invariant.

DEFINITION. *Let* $d_n: R^n \times R^n \to R$ *be given by*

$$d_n(x, y) = \begin{cases} 0 & \text{if } x = y \\ a & \text{if } \|x\| = \|y\| \quad \text{but} \quad x \neq y \\ b & \text{if } \|x\| \neq \|y\| \end{cases} \tag{3}$$

where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ *and* a *and* b *are positive real numbers such that* $a \leqq 2b$.

THEOREM 4. *The function defined in* (3) *is a metric on* R^n *which is rotation invariant but not* (*unless* $a = b$) *translation invariant.*

We omit the easy proof of the theorem but note that the condition $a \leqq 2b$ is essential. The relative magnitudes of a and b are subject only to this condition. (Compare with Theorem 1.) In what follows we assume that $a \neq b$ and indicate a few interesting properties of d_n.

The metric (3) treats points on the same circle (center origin) to be at distance a while others are at distance b. In terms of the road map interpretation in [1] this means, for $n=2$, that if the fuel tank capacity is sufficient to take us a distance of max $\{a, b\}$ units then we can go from anywhere to anywhere without stopping for fuel. If, however, $a<b$ and the tank capacity is a, we can only go on a circle around the origin no matter how many refuellings are made; that is, we cannot leave that orbit. If the tank capacity is b and $b<a$ we can, starting from x, go anywhere *except* to points on the circle with radius $\|x\|$ and center at the origin. It is interesting to note in this last case that we can *approach* these exceptional places arbitrarily closely (in terms of the usual distance function!). Of course one refuelling makes all points accessible.

We enumerate below (without proof) a few other properties of the metric which are easily verified. For simplicity we consider only the case $n=2$, and define the area of a circle of radius r to be πr^2 and the area of a rectangle to be the product of its length and width. (i) The area of any circle with center at origin is πb^2. For other circles it is undefined. (ii) The area of a rectangle is a^2 or ab according as four or two of its vertices belong to some circle with center at origin. Otherwise the area is b^2. (iii) The area of a rectangle may be strictly smaller than that of a subrectangle. (iv) The area of a union of disjoint rectangles is the sum of its component rectangles. Here two rectangles with a common edge are not considered disjoint.

Reference

1. S. K. Hildebrand and H. W. Milnes, An interesting metric space, this MAGAZINE, 41 (1968) 244–247.

ANSWERS

A472. If $\alpha \geqq \pi/4$ then $\beta \leqq \pi/4 \leqq \alpha$, and $2+\tan\alpha+\tan\beta \leqq 2+2\tan\alpha$. But $\sec\alpha = 1/\cos\alpha \geqq \sin\alpha/\cos\alpha = \tan\alpha$, and $\sec\beta \geqq 1$ so $2\sec\alpha+2\sec\beta \geqq 2\tan\alpha+2$. A similar proof holds for $\beta \geqq \pi/4$. Assume $\alpha \leqq \pi/4$ and $\beta \geqq \pi/4$. Then $2+\tan\alpha+\tan\beta \leqq 4 \leqq 2\sec\alpha+2\sec\beta$.

A473. No, it does not follow. Simply let $d(x, y) = \min(1, |x-y|)$, and note that this metric generates the usual topology in R, but $d(0, x) = d(0, y) = 1$ if $1<x<y$.

A474. The area of the annular space is 144π which, to four decimal places, is $452\cdot 3893$. Thus 3893 452 represents *CHIC DEB*.

A475. Solution: $x^3 - 117y^3 \equiv 0, \pm 1 \pmod 9$ and hence cannot equal five.

A476. $z^2 = z\cdot z$ implies that $z=\sqrt{z\cdot z}$ but $z=\sqrt{z\cdot z}$ implies $z=\sqrt{z\sqrt{z\cdot z}}$ and so on.

(Quickies on page 112.)

ON e AND ITS APPROXIMATION

C. I. LUBIN and A. J. MACINTYRE

The use of the equation $1=\int_1^e dx/x$ to define and approximate e (as in Hardy [1] p. 405 et seq., Taylor [2] p. 307, among others) can be extended to discussing the limit, $\lim_{n\to\infty}(1+1/n)^n$, and to finding a closer approximation to e.

Existence of the limit. The proof of the existence of the limit is indicated by using the monotonic increasing, continuous function

$$F(u)=\int_1^u \frac{dx}{x} \qquad 0<u<\infty \tag{1}$$

which has a unique continuous inverse and for which $F(u^n)=nF(u)$ (see Hardy l.c.). To avoid the introduction of irrational exponents the number n will be restricted to rational values. Let us write $w_n=(1+1/n)^n$ and $v_n=F(w_n)$ and thus $v_n=n\int_1^{1+1/n} dx/x$. Since the integrand is a decreasing function of x we have

$$n\left(\frac{1}{1+1/n}\right)(1/n)<v_n<n(1/n) \quad \text{or} \quad 1/(1+1/n)<v_n<1.$$

Thus we can immediately infer the existence of $\lim_{n\to\infty}F(w_n)$ and conclude that its value is 1. Then clearly the continuous inverse w_n of $F(w_n)$ as $n\to\infty$ approaches a limit also, which is designated as e (Hardy, l.c.).

Approximation to e. In the references just cited, elementary considerations of the equation (1), immediately lead to the result $2.56<e<3.00$. For a closer approximation for e, we write $\gamma=e^{1/k}$ where k is a positive integer and seek an approximation for γ. We have

$$1/k=\int_1^\gamma \frac{dx}{x}\,. \tag{2}$$

Making the change of variable, $x=(\gamma+1)(v+1)/2$ and putting

$$\alpha=(\gamma-1)/(\gamma+1) \qquad 0<\alpha<1 \tag{3}$$

in (2) we are led to the equivalent expression,

$$1/k=2\int_0^\alpha \frac{dv}{1-v^2}\,. \tag{4}$$

Our problem now is to find an approximation to α^*, the unique real root of equation (4) on the interval $0<\alpha<1$.

Writing the integrand as $1/(1-v^2)=1+v^2+v^4+v^6/(1-v^2)$ equation (4) becomes

$$1/2k=\alpha+\alpha^3/3+\alpha^5/5+\int_0^\alpha \frac{v^6}{1-v^2}\,dv.$$

An approximation to the root α^* of this equation can be found by obtaining, approximately, the real zero of the function

$$H(t) = -1/2k + t + t^3/3 + t^5/5.$$

Take $1/2k$ as a first approximation to the zero, then Newton's method gives as a second approximation $\alpha_1 = 1/2k - \beta$, where the term β is given by

$$(5) \qquad \beta = H(1/2k)/H'(1/2k) = \frac{(1/2k)^3/3 + (1/2k)^5/5}{1 + (1/2k)^2 + (1/2k)^4}.$$

Thus we have for α_1,

$$\alpha_1 = (120k^4 + 20k^2 + 6)/(240k^5 + 60k^3 + 15k).$$

For a numerical result, pick for k, say the value 8, then the above approximation yields

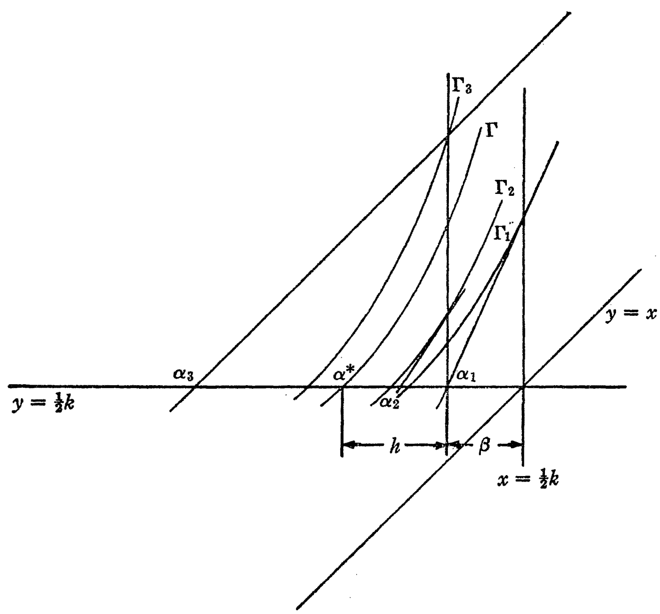
$$\alpha_1 = 246{,}403/3{,}947{,}580.$$

Using (3), we get

$$\gamma_1 = (1 + \alpha_1)/(1 - \alpha_1) = 4{,}193{,}983/3{,}701{,}177$$
$$= 1.133148455.$$

Thus we have an an approximation for e

$$\gamma_1^8 = 1.133148455^8 = 2.71828186.$$



Bounds for the error. The above estimate α_1, and our bounds for the error depend on replacing the curve Γ, $y = \int_0^x dv/1 - v^2$, by other curves and finally by straight lines. (See the accompanying diagram.) All the curves and straight lines run at very nearly 45° to the x, y axes with all slopes greater than 1. We consider the curves, $\Gamma_1: y=x+x^3/3+x^5/5$ and $\Gamma_2: y=x+x^3/3+x^5/5+x^7/7$. Evidently curve Γ_1 lies below Γ_2 and both lie below the curve Γ. We also consider the curve $\Gamma_3: y=x+x^3/3+x^5/5+x^7/7(1-x^2)$ which evidently lies above the curve Γ on the interval $0<x\leqq 1/2k$.

The points where these curves cross the line $y=1/2k$ lie progressively further to the left. Designate the abscissa of the point of intersection of the curve Γ with this line by α^* but for the other curves it is unnecessary to find or designate their intersections with this line.

We introduce, (Newton's method), the tangent line to Γ_1 at $x=1/2k$ which intersects $y=1/2k$ further to the right of $x=\alpha^*$, at the point which gives us the above estimate α_1; the tangent line to Γ_2 at $x=\alpha_1$ which intersects $y=1/2k$ at $x=\alpha_2$, also to the right; and finally a line at 45° to the axes through the point on Γ_3 with $x=\alpha_1$ which intersects the line $y=1/2k$ further to the left at $x=\alpha_3$. Thus the approximation α_1 is in excess so the difference $(\alpha_1-\alpha_2)$ will lead to an improved approximation for α^*, still in excess, while the difference $(\alpha_1-\alpha_3)$ leads to a bound below.

Computation gives us

$$\alpha_1-\alpha_2=(\alpha_1+\alpha_1^3/3+\alpha_1^5/5+\alpha^7/7-1/2k)/(1+\alpha_1^2+\alpha_1^4+\alpha_1^6)$$

$$>(\alpha_1+\alpha_1^3/3+\alpha_1^5/5+\alpha_1^7/7-1/2k)(1-\alpha_1^2)$$

$$\alpha_1-\alpha_3=\alpha_1+\alpha_1^3/3+\alpha_1^5/5+\alpha_1^7/7(1-\alpha_1^2)-1/2k.$$

Writing $h=(\alpha_1-\alpha^*)>0$, we see

$$\alpha_1-\alpha_2<h<\alpha_1-\alpha_3$$

or $(\alpha_1+\alpha_1^3/3+\alpha_1^5/5+\alpha_1^7/7-1/2k)(1-\alpha_1^2)<h<(\alpha_1-\alpha_3)$. We use the binomial inequalities on the powers of $\alpha_1^n=(1-2k\beta)^n/(2k)^n$, i.e., $1/(2k)^n-n\beta/(2k)^{n-1}<\alpha_1^n<1/(2k)^n-n\beta/(2k)^{n-1}+n(n-1)\beta^2/2(2k)^{n-2}$, and the value of β in (5) and obtain the result

$$h''=\alpha_1^7(1-\alpha_1^2)/7<h<\beta^2(1/2k+2/(2k)^3)+\alpha_1^7/7(1-\alpha_1^2)=h'. \tag{6}$$

Turn next to the corresponding error in γ_1, $g=\gamma_1-\gamma^*$ where $\gamma^*=(1+\alpha^*)/(1-\alpha^*)$ and $\gamma_1=(1+\alpha_1)/(1-\alpha_1)$. Thus we have $g=2h/(1-\alpha_1)(1-\alpha_1+h)$ and $2h<g<2h/(1-\alpha_1)^2$.

Using (6), we find as bounds for g

$$g''=2h''<g<2h'/(1-\alpha_1)^2=g'.$$

Finally for the error $f=\gamma_1^k-(\gamma_1-g)^k$ arising from the use of γ_1^k as approximation for $e=\gamma_1^{*k}$, on again using the binomial inequalities, we are led to

$$k\gamma_1^{k-1}g''-k(k-1)\gamma_1^{k-2}g''^2/2<f<k\gamma_1^{k-1}g'.$$

For the particular numerical approximation to e above, with $k=8$, we have

$$\alpha_1=0.06241874769 \qquad b=1-\alpha_1=0.00008125231$$

$$\gamma_1 = (1+\alpha_1)/(1-\alpha_1) = 1.1331484552$$

$$5.253 \quad 10^{-10} < h < 9.4526 \quad 10^{-10}$$

$$1.0505 \quad 10^{-9} < g < 2.1507 \quad 10^{-9}$$

$$2.0159 \quad 10^{-8} < f < 4.1275 \quad 10^{-8}$$

whence

$$2.71828182 < e < 2.71828184.$$

References

1. G. H. Hardy, A Course of Pure Mathematics, 10th ed., Cambridge University Press, England, 1952.

2. A. E. Taylor, Calculus with Analytic Geometry, Prentice-Hall, Englewood Cliffs, New Jersey, 1959.

A DIRECT PROOF OF THE STEINER-LEHMUS THEOREM

JOVAN V. MALEŠEVIČ, Šabac, Yugoslavia

The above theorem is: *If the bisectors of the two internal angles of a triangle are equal, then the triangle is isosceles.*

In this article a direct proof of this theorem will be given. The author has found no such proof in the literature.

Let ABC be a triangle and let $\overline{AD} = \overline{BE}$, where AD and BE are the bisectors of the internal angles $\angle CAB$ and $\angle CBA$, respectively (see Figure 1).

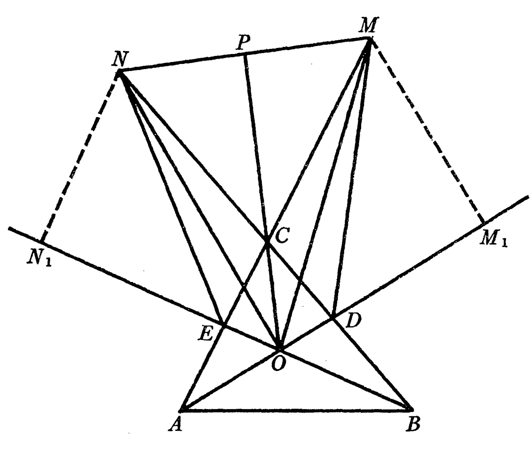


FIG. 1.

Produce the sides AC and BC beyond the vertex C to the points M and N, respectively, such that $\overline{CM} = \overline{CN} = \overline{AB}$. Then the bisector OC of the internal

$$\gamma_1 = (1+\alpha_1)/(1-\alpha_1) = 1.1331484552$$

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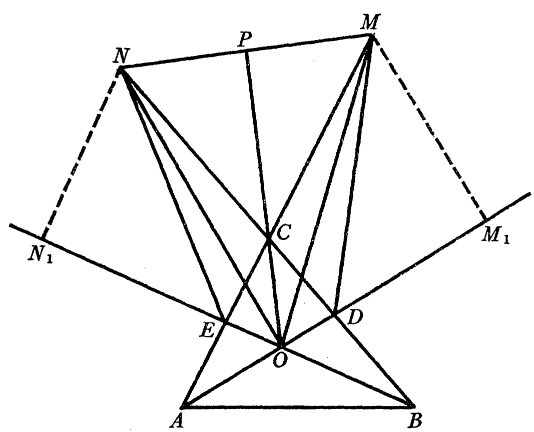


FIG. 1.

Produce the sides AC and BC beyond the vertex C to the points M and N, respectively, such that $\overline{CM} = \overline{CN} = \overline{AB}$. Then the bisector OC of the internal angle $\angle ACB$ meets the straight line MN in P. Since

$$OP \perp MN, \qquad \overline{NP} = \overline{PM}.$$

it follows that

$$\angle PON = \angle POM, \qquad \overline{ON} = \overline{OM}.$$

From the fact that area $\triangle ABE$ = area $\triangle ECN$ and area $\triangle ABD$ = area $\triangle DCM$ it follows that area $\triangle ADM$ = area $\triangle BEN$ (= area $\triangle ABC$) and one obtains, if $NN_1 \perp BE$ and $MM_1 \perp AD$, $\overline{NN_1} = \overline{MM_1}$ which implies that $\angle EON = \angle DOM$ and hence

$$\angle EOC = \angle DOC, \qquad \angle CAB = \angle CBA$$

which was to be proved.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College
ASSOCIATE EDITOR, MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be legible and submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

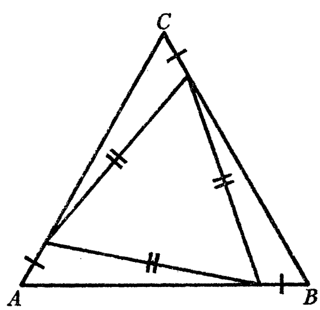
Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 *Fulton Avenue, Van Nuys, California* 91401.

To be considered for publication, solutions should be mailed before August 1, 1970.

PROPOSALS

754.* *Proposed by NSF Class at University of California at Berkeley.*

Show that the triangle ABC is equilateral.



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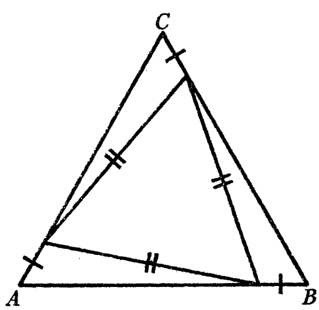


755. *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Let $f(x)=x^2+px+q$ where p and q are real numbers. Show that at least one of the numbers $|f(1)|$, $|f(2)|$, or $|f(3)|$ is not less than one-half. Generalize to a polynomial of degree n.

756. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine closed and centrally symmetric curves C, other than circles, such that the product of two perpendicular radius vectors (issued from the center) be a constant.

757. *Proposed by Erwin Just, Bronx Community College, New York.*

Let p and q be primes and x an integer such that

$$p \,\Big|\, \sum_{k=0}^{q^r-1} x^k, \qquad (r \geqq 1).$$

Prove that either $p-1$ is divisible by q or $p=q$.

758. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

If the internal bisector of angle A of a triangle ABC is perpendicular to the line joining the incenter and the orthocenter, show that $\cos B+\cos C=1$.

759.* *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Circles A, C, and B with radii of lengths a, c, and b, respectively, are in a row, each tangent to a straight line DE. Circle C is tangent to circles A and B. A fourth circle is tangent to each of these three circles. Find the radius of the fourth circle.

760. *Proposed by Hugh M. Edgar, Chris W. Avery, and Albert E. Pollatchek, San Jose State College, California.*

Characterize those positive integral values of n and m for which the nth cyclotomic field is contained in the mth cyclotomic field.

SOLUTIONS

Late Solutions

Andrew N. Aheart, West Virginia State College: **725**; *Donald Chand, Lockheed-Georgia Company:* **725, 729**; *Charles Chouteau, West Virginia State College:* **726**; *J. R. Kuttler, Johns Hopkins University:* **725, 729, 730**; *Gerald C. Dodds, HRB-Singer Company:* **732**; *Philip Haverstick, Ft. Belvoir, Virginia:* **726, 732**; *Peter A. Lindstrom, Genesee Community College:* **726**; *Bernard J. Portz, Jesuit College:* **730**; *Henry J. Ricardo, Yeshiva University:* **725**; *John W. Schleusner, West Virginia University:* **728, 729**; *E. F. Schmeichel, College of Wooster:* **724, 731**; *Charles W. Trigg, San Diego, California:* **726, 730**; *R. F. Wardrop, Central Michigan University:* **726.**

Erratum

The last line of the answer A402 on Page 203 of the September, 1969, issue, should read " . . . $f(x_1)=f(x_2)$ . . . " instead of " . . . $f(x)=f(y)$ . . . "

Rats Eat Tea

733. [September, 1969] *Proposed by Richard L. Breisch, University of Colorado.*

Solve this determinant equation with nonnegative digits: ($R\neq 0$)

$$R\quad A\quad T\quad S = -\begin{vmatrix} E & A & T \\ A & T & E \\ T & E & A \end{vmatrix}.$$

Solution by Rosalie Farrand, Castilleja High School, Palo Alto, California.

Assume first that $RATS$ does not refer to a product of the four digits, but does mean $1000R+100A+10T+S$.

Using the usual rules for evaluating a determinant, we find

$$-\begin{vmatrix} E & A & T \\ A & T & E \\ T & E & A \end{vmatrix} = E^3 + A^3 + T^3 - 3(E\cdot A\cdot T),$$

where $(E\cdot A\cdot T)$ refers to the product of the three digits. It is then easy to conclude that one of the digits must be zero to eliminate the term $-3(E\cdot A\cdot T)$, and that the remaining two digits of the determinant must be 7 and 9. We then have this result.

$$1072 = -\begin{vmatrix} 9 & 0 & 7 \\ 0 & 7 & 9 \\ 7 & 9 & 0 \end{vmatrix}, \quad \text{so that } R = 1,\ A = 0,\ T = 7,$$

$S=2$, and $E=9$.

Also solved by Merrill Barnebey, Wisconsin State University at LaCrosse; Wray G. Brady, Slippery Rock State College, Pennsylvania; Laurence Beller, Bronx, New York; Bruce A. Broemser, El Sobrante, California; Philip Cheifetz, Nassau Community College, New York; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; Jay M. Fleisher, Carnegie-Mellon University; Sandra V. Foster, Long Island University; Myron A. Fribush, Bangor, Pennsylvania; B. L. Gingrich, Endicott, New York; Michael Goldberg, Washington, D.C.; Philip Haverstick, Ft. Belvoir, Virginia; Alfred Kohler, Long Island University; Otto Mond, Suffern, New York; Richard F. Morrow, Dallas, Texas; George A. Novacky, Jr., St. Anselm High School, Swissvale, Pennsylvania; Mary A. Ratchford, Mt. Saint Mary College, Newburgh, New York; Steve Rohde, Lehigh University; E. F. Schmeichel, College of Wooster, Ohio; W. A. Schmidt, Texas A and M University; Donval R. Simpson, Fairbanks, Alaska; E. P. Starke, Plainfield, New Jersey; David J. Stevenson, West Virginia State College; John H. Tiner, High Ridge, Missouri; Charles W. Trigg, San Diego, California; Zalman Usiskin, University of Chicago; R. F. Wardrop, Central Michigan University; Kenneth M. Wilke, Topeka, Kansas; and the proposer. One incorrect solution was received.

A Convex Curve

734. [September, 1969] *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let K be a plane convex curve which can rotate in an equilateral triangle of altitude h so as to be always tangent to its three sides. Prove that:

a) If $h > 2\pi/(2\pi - 3\sqrt{3})$, then K always encloses a lattice point.

b) If $h < 1 + \sqrt{3}/2$, then one can rotate and translate the curve K in the plane so that in one position at least it will contain no lattice points.

Solution by Michael Goldberg, Washington, D. C.

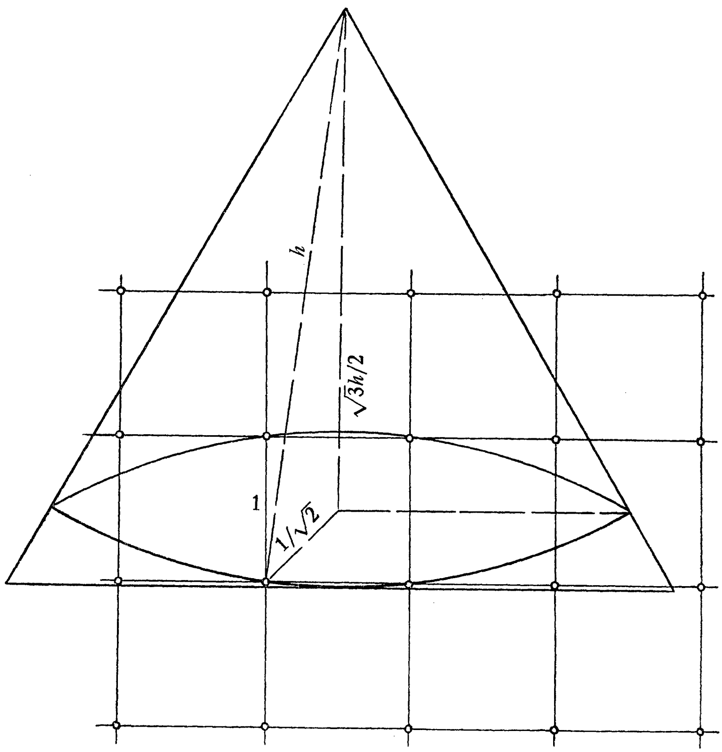
Both limits can be improved.

(a) Of all the rotors in an equilateral triangle, the one which has the smallest inscribed square is the one made of two 60° arcs of radius h. Then, if the square has unit edge, the height h satisfies the following equation:

$$h^2 = (\sqrt{3}h/2)^2 + (1/\sqrt{2})^2 + 2(\sqrt{3}h/2)(1/\sqrt{2})/\sqrt{2}$$
$$= 3h^2/4 + 1/2 + \sqrt{3}h/2$$

or $h^2 - 2\sqrt{3}h - 2 = 0$, from which

$$h = (2\sqrt{3} + \sqrt{12 + 8})/2 = \sqrt{3} + \sqrt{5} \approx 3.968 < 2\pi/(2\pi - 3\sqrt{3}) \approx 5.78.$$



(a) $h = \sqrt{3} + \sqrt{5} \approx 3.968$

Hence, if $h > \sqrt{3} + \sqrt{5}$, the rotor K will contain a lattice point when placed anywhere in the plane.

(b) If the height of the equilateral triangle is h, then the radius of its inscribed circle is $h/3$. The square inscribed in this circle has edge $\sqrt{2}h/3$. If the edge is taken as unity, then $h=3/\sqrt{2}\approx 2.14>1+\sqrt{3}/2$. For other rotors, the value of h would be larger. If $h<3/\sqrt{2}$, then the circle K can be placed so that it encloses no lattice point.

Also solved by E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

A Partition of Triangular Numbers

735. [September, 1969] *Proposed by Charles W. Trigg, San Diego, California.*

Find a triangular number which can be partitioned into three 3-digit primes which together contain the nine positive digits.

Solution by Donald Beane and E. F. Schmeichel (jointly), College of Wooster, Ohio.

Observe that a triangular number must end in 0, 1, 3, 5, 6, or 8. Since a 3-digit prime ends in 1, 3, 7, or 9, the only possible digits in the units position of the sum of three primes using all the positive integers are 1 (if the primes end in 1, 3, and 7), 3 (if the primes end in 1, 3, and 9), 7 (if the primes end in 1, 7, and 9) and 9 (if the primes end in 3, 7, and 9). Thus only the triangular numbers ending in 1 or 3 need to be considered. We observe that in either of these cases, one of the primes must end in 1 and one must end in 3.

Moreover, the sum S of three 3-digit numbers using all positive digits satisfies $774\leqq S\leqq 2556$. The only triangular numbers in this range which end in 1 or 3 are 861, 903, 1431, 1653, 1711, 1891, 1953, and 2211. Noting that 1 and 3 must occur in the units position of two of the three primes to be summed, we find that the smallest numbers that could appear in the hundreds position of the three primes are 2, 4, and 5. Since $2+4+5>9$, we eliminate 861 and 903 as possibilities.

We now partition the set of three digit primes without repeated digits into three parts: those ending in 1, those ending in 3, and those ending in 7 or 9. (The list in each part could be shortened, of course, by eliminating primes which contain 0, primes which end in 1 but contain a 3, and so on.)

We then determine all possible sums which arise by selecting one number from each group. For this, a Control Data Corporation 6400 was used. The addition was printed out only when the sum was one of our six candidates. Total computer time was 1.5 seconds, and we obtained the following results:

$$1431=\sum_{i=1}^{53} i = 283+457+691 = 257+491+683$$

$$1953=\sum_{i=1}^{62} i = 421+673+859 = 479+653+821$$

Also solved by Gerald C. Dodds, HRB-Singer, State College, Pennsylvania; Alfred Kohler, Long Island University, New York; E. P. Starke, Plainfield, New Jersey, and the proposer.

Partial solutions ($1\leqq S<4$) *were obtained by Richard L. Breisch, University of Colorado; Philip Haverstick, Ft. Belvoir, Virginia; Frank L. Kerr, Napa, California; Melvin D. Rein, Morton, Illinois; and Kenneth M. Wilke, Topeka, Kansas. One incorrect solution was received.*

Kohler found among the set of triangular numbers the number 666 and wondered if others had noted that the mark of the beast is a triangular number. Moreover, 666 is the 36th triangular number and $36 = 6 \cdot 6$.

A Straight Line is the Shortest . . .

736. [September, 1969] *Proposed by Alfred Kohler, Long Island University.*

Al, Bill Chuck, and Don all live in the same school district. When Al faces the school from his home, Bill's home is directly to Al's right. Bill lives directly to the west of school, and Chuck lives due south of Bill. At his home, Don can see the sun setting behind Chuck's house at times.

Al's home is as far from Bill's as it is from school, while Chuck lives twice as far from Bill as Bill does from school. Don lives three times as far from Chuck as Chuck does from school, and Don lives ten times as far from school as does Al.

Show that Al's home, Don's home, and the school are collinear.

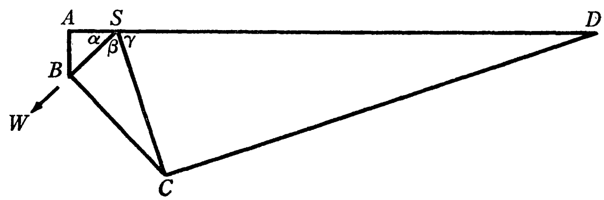


FIG. 1

I. *Solution by C. D. O'Shaughnessy, University of Saskatchewan.*

Labelling their houses A, B, C, D and the school S, one has a figure such as Figure 1. All locations are straightforward, except for that of D. Since Don can sometimes see the sunset behind Chuck's house, the location of Don's house to the east of Chuck's follows.

If distances are as labelled with $AS=x$, $SB=y$, and $SC=z$, then considering right angled triangles we have

$$\begin{aligned} y^2 &= 2x^2 \\ z^2 &= 5y^2 \\ &= 10x^2 \end{aligned}$$

We must then show that

$$\angle ASD = 180°.$$

Now,

$$\angle ASB = 45° \qquad \text{since } \Delta ASB \text{ is isosceles;}$$

$$\angle BSC = \tan^{-1}\frac{2y}{y} = \tan^{-1} 2.$$

As for $\angle CSD$, we first note that

$$(3z)^2 + z^2 = 10z^2 = 100\,x^2$$

so that ΔCSD is right angled. Then

$$\angle CSD = \tan^{-1}\frac{3z}{z} = \tan^{-1} 3.$$

Therefore,

$$\begin{aligned}\angle ASD &= \angle ASB + \angle BSC + \angle CSD \\ &= 45^\circ + \tan^{-1} 2 + \tan^{-1} 3 \\ &= 45^\circ + \tan^{-1}\left(\frac{2+3}{1-(2)(3)}\right) \\ &= 45^\circ + 135^\circ = 180^\circ\end{aligned}$$

where we use the formula

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

to evaluate $\theta+\phi = 135°$.

II. *Solution by George F. Corliss, Michigan State University.*

Denote Al's house by A, Bill's by B, Chuck's by C, Don's by D, and the school by S. Let $S(0, 0)$, $AS=1$, and west be the negative x direction. Then $B(-\sqrt{2}, 0)$, $A(-1/\sqrt{2}, 1/\sqrt{2})$, $C(-\sqrt{2}, -2\sqrt{2})$. $CD=3\sqrt{10}$ and $DS=10$. Then $D=(5\sqrt{2}, -5\sqrt{2})$ or $(-7\sqrt{2}, \sqrt{2})$. Since Don sees the sun set behind Chuck's house at times, he must live to the east of Chuck, or at $(5\sqrt{2}, -5\sqrt{2})$. Then A, S, and D all lie on the line $y=-x$.

Also solved by William J. Amadio, Brooklyn College; Henry F. Burch, Edwards Air Force Base, California; Michael Goldberg, Washington, D. C.; Frank L. Kerr, Napa, California; David J. Lohuis, Bucknell University; Charles McCracken, Florida Southern College; Joseph E. Mueller, Bloomsburg State College, Pennsylvania; Simeon Reich, Israel Institute of Technology, Haifa, Israel; E. F. Schmeichel, College of Wooster, Ohio; W. A. Schmidt, Texas A and M University; Paul Sugarman, Massachusetts Institute of Technology; Charles W. Trigg, San Diego, California; Zalman Usiskin, University of Chicago; and the proposer.

The proposer indicated that he developed the problem to let the solver discover the pretty relationship arctan 1+arctan 2+ arctan 3 = 0.

A Perceptive Hippie

737. [September, 1969] *Proposed by Irving A. Dodes, Kingsborough Community College, Brooklyn, New York.*

Mr. S. D. S. Hippie, inveterate seeker after truth (far after), awoke from a slight doze to hear his plane geometry teacher remark that if the midpoints of any quadrilateral are joined, the result is a parallelogram. Not to be outdone in untenable hypotheses, Mr. Hippie opined that if trisection points are joined, the result will also be a parallelogram.

Prove that no other dividing points independent of the lengths of the sides exist to produce a parallelogram.

Solution by Alfred Kohler, Long Island University, New York.

The problem is poorly stated. What the proposer seems to be saying is patently false, yet it is difficult to attribute any other meaning to the problem. Does the proposer mean to imply that the midpoints, trisection points, and n-section points in general of a straight line segment are independent of the length of the segment? They are and they aren't, depending on the meaning assigned to the word "independent."

The following theorem is well known: *Given a triangle ABC and any two points D and E on AB and AC, respectively, together with the straight line joining D and E. Then DE is parallel to BC if and only if* $AB/AD=AC/AE$.

The simple corollary of this theorem is that if the n-section points of the sides of a quadrilateral are joined in a suitable manner, then $n-1$ different parallelograms inscribed inside the quadrilateral are obtained. A unique parallelogram is of course obtained only in the case of midpoints ($n=2$).

Pehaps the proposer wishes to point out that the n-section points to be joined may not be chosen at random, but that, of course, is obvious. The lines joining n-section points on adjacent sides of the quadrilateral must divide the two sides in the same proportion. The preceding theorem assures one that such lines will be parallel to the diagonals of the quadrilateral, and will therefore also be parallel to each other.

Also solved by Michael Goldberg, Washington, D. C.; Zalman Usiskin, University of Chicago; and Sam Zaslavsky, City University of New York.

The following showed that no solution other than midpoints exist if the n-section points are taken in order proceeding around the quadrilateral: *J. C. Binz, Bern, Switzerland; Bill Knight, University of Wyoming; Ray B.Robinson, Butler, Tennessee; E. F. Schmeichel, College of Wooster, Ohio; Charles W. Trigg, San Diego, California; and the proposer.*

A Very Narrow River

738. [September, 1969] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

There is a river with parallel and straight shores. A is located on one shore and B on the other, with $AB=72$ miles. A ferry boat travels the straight path AB from A to B in four hours and from B to A in nine hours. If the boat's speed on still water is $v=13$ mph, what is the velocity of the flow?

Solution by Bill Knight, University of Wyoming.

Since the time of travel is not the same for both directions, B is not directly opposite A, but is some distance down the river. The 72 miles from A to B were traveled in 4 hours, an average of 18 mph, while the return trip averaged only 8 mph. Let the current be s.

For the average of the trip down the river to be 18 mph, the vector sum of the ferry and the current must be 18 mph. Since the magnitude of a vector sum is less than or equal to the sum of the magnitudes of the vectors, it is clear that

$$13 \text{ mph} + s \geqq 18 \text{ mph}, \quad \text{and thus}$$
$$s \geqq 5 \text{ mph}$$

Returning from B to A, the maximum possible magnitude for the vector sum is 13 mph$-s$. Thus,

$$13 \text{ mph} - s \geqq 8 \text{ mph}$$
$$s \leqq 5 \text{ mph}.$$

Since $s \geqq 5$ mph and $s \leqq 5$ mph, $s = 5$ mph.

An alternate method would be to allow the width of the river to be m miles and find s for any m. It is possible to find m using this method. It involves solving the following equation:

$$13\sqrt{5184 - m^2} = 4\sqrt{13689 - m^2} + 9\sqrt{2704 - m^2}.$$

The only solution is $m = 0$ mph. With this value of m it follows that $s = 5$ mph.

Also solved by Richard L. Breisch, University of Colorado; Bruce A. Broemser, El Sobrante, California; Raphael T. Coffman, Richland, Washington; George F. Corliss, Michigan State University; Mickey Dargitz, Ferris State College, Michigan; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; Frank Eccles, Phillips Academy, Massachusetts; W. W. Funkenbusch, Michigan Technological University; Michael Goldberg, Washington, D. C.; John M. Howell, Littlerock, California; Alfred Kohler, Long Island University, New York; Lew Kowarski, Morgan State College, Maryland; J. R. Kuttler, Johns Hopkins University; Joseph O'Rourke, St. Joseph's College, Pennsylvania; C. D. O'Shaughnessy, University of Saskatchewan; John E. Prussing, University of Illinois; John R. Ray, Clemson University; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Ray B. Robinson, Butler, Tennessee; Steve Rohde, Lehigh University; E. F. Schmeichel, College of Wooster, Ohio; W. A. Schmidt, Texas A and M University; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; A. W. Walker, Toronto, Canada; Sam Zaslavsky, City University of New York; and the proposer.

Property of Central Conics

739. [September, 1969] *Proposed by Mehmet Agargun, Diyarbakir, Turkey.*

Let the vertical projections of a variable point P of a central conic with center O, on the major and minor axes be C and S, and the tangent line at P intersects these axes at C' and S' respectively. Then prove that:

$$\overline{OC}\cdot\overline{OC'} - \overline{OS}\cdot\overline{OS'} = \text{constant}.$$

Solution by P. D. Thomas, Naval Research Laboratory, Washington, D. C.

If the central conic has the equation $ax^2+by^2-1=0$, then the tangent at $P(x_1, y_1)$ on the conic has for equation $axx_1+byy_1-1=0$ with x and y intercepts $\overline{OC'}=1/ax_1$, $\overline{OS'}=1/by_1$ respectively. Since $\overline{OC}=x_1$, $\overline{OS}=y_1$, we have $\overline{OC}\cdot\overline{OC'}=1/a$, $\overline{OS}\cdot\overline{OS'}=1/b$, the difference being $1/a-1/b$.

Also solved by Santo M. Diano, Philadelphia, Pennsylvania; Michael Goldberg, Washington, D. C.; Philip Haverstick, Ft. Belvoir, Virginia; Lew Kowarski, Morgan State College, Maryland; C. Stanley Ogilvy, Hamilton College, New York; Simeon Reich, Israel Institute of Technology, Haifa, *Israel; Steve Rohde, Lehigh, University; E. F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; A. W. Walker, Toronto, Canada; Sam Zaslavsky, City University of New York; and the proposer.*

Comment on Problem 716

716. [January and September, 1968] *Proposed by John M. Howell, Los Angeles City College.*

Suppose that we have a deck of cards numbered 1, 2, $\cdots n$ and a second deck numbered 1, 2, $\cdots (n+d)$, $d \geqq 0$, and that we shuffle the two decks and draw one card at a time from each until the smaller deck is exhausted. Find the probability of r matching draws and the mean and variance of the number of matching draws.

Comment by Frederick Mosteller, Harvard University.

(1) The probability is given in an incomplete form for $d=0$ in the formula at the bottom of Page 219 because the cases $r=n$ and $r=n-1$ are not properly included (after the erroneous coefficient $n!$ is deleted). We need

$$\{1/0! - 1/1! + 1/2! \cdots + (-1)^{n-r}/(n-r)!\}/r!.$$

The cancellation of the first two terms in the published version spoils the full generality of the formula.

(2) The mean and variance for $d=0$ are 1 and 1, suggesting the near Poissonness of the distribution for large n and small r indicated by the correct form $u_{n,r}$.

(3) For general n and d the mean and variance of the number of matches are

$$\mu_{n,d} = \frac{n}{n+d}, \qquad \sigma^2_{n,d} = \frac{n}{n+d}\left[1 - \frac{d}{(n+d)(n+d-1)}\right],$$

again suggesting approximate Poissonness when $d/(n+d)(n+d-1) \ll 1$.

A similar remark was submitted by E. F. Schmeichel, College of Wooster, Ohio.

Comment on Q411

Q411. [May, 1967; May, 1969] Show that the sum of two successive odd primes is the product of at least three (not necessarily distinct) prime factors.

[*Submitted by John D. Baum*]

Comment by Charles W. Trigg, San Diego, California.

As is evident from the reference (but not the text) of the comment on Page 163 of the May, 1969, issue, that argument applies only to twin primes. However, these are not the only consecutive primes with a sum divisible by 6.

Consider the number (which include all the odd primes but 3) with the forms $6k \mp 1$.

If two of these numbers have different forms, then their sum, $S=(6k_1-1)+(6k_2+1)=6(k_1+k_2)$. Clearly, this has at least three prime factors. Twin primes ($k_1=k_2$) fall in this category, as do other consecutive primes with $k_1 \neq k_2$ such as 401 and 409.

If the numbers have like forms, $S=(6k_1 \pm 1)+(6k_2 \pm 1)=2[3(k_1+k_2) \pm 1]$. Then if k_1 and k_2 are of opposite parity, the quantity in the bracket is even, so S has at least three prime factors. If k_1 and k_2 are of like parity, as is the case with $1307+1319=2(13)(101)$ and $1531+1543=2(29)(53)$, the analysis is not easy. For consecutive primes, recourse is taken to the compact, elegant proof in the original Quickie.

An example of two odd numbers with k's of like parity whose sum has only *two* prime factors is $7+19=2(13)$.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q472. Let $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha+\beta \leqq \pi/2$. Prove

$$2+\tan \alpha+\tan \beta \leqq 2 \sec \alpha+2 \sec \beta.$$

[*Submitted by Harley Flanders*]

Q473. Let d be a metric which generates the usual topology in the real line. Does it follow that $d(0, x) \neq d(0, y)$ if $0<x<y$ are real numbers?

[*Submitted by E. F. Schmeichel*]

Q474. Two concentric circles are of such a size that a chord of the larger which is tangent to the smaller is exactly two feet long. The area of the annular space (in sq. in.) is carried out to four decimal places and it is observed that when converted into letters ($A=1$, $B=2$, etc.) the decimal and integral parts might represent a certain type of well-dressed young lady. Who is she?

[*Submitted by Walter Penney*]

Q475. On Page 26, Volume 6, MAA Studies in Mathematics—*Studies in Number Theory*, there appears this statement:

"The equation $x^3-117y^3=5$ is known to have at most 18 integral solutions but the exact number is unknown." Prove, in fact, that the equation has no integral solutions.

[*Submitted by Graham C. Thompson*]

Q476. Show that

$$z = \sqrt{z\sqrt{z\sqrt{z\sqrt{z}}}} \cdots$$

[*Submitted by P. G. Pantelidakis*]

(Answers on page 97.)

We omit the easy proof of the theorem but note that the condition $a \leqq 2b$ is essential. The relative magnitudes of a and b are subject only to this condition. (Compare with Theorem 1.) In what follows we assume that $a \neq b$ and indicate a few interesting properties of d_n.

The metric (3) treats points on the same circle (center origin) to be at distance a while others are at distance b. In terms of the road map interpretation in [1] this means, for $n=2$, that if the fuel tank capacity is sufficient to take us a distance of max $\{a, b\}$ units then we can go from anywhere to anywhere without stopping for fuel. If, however, $a<b$ and the tank capacity is a, we can only go on a circle around the origin no matter how many refuellings are made; that is, we cannot leave that orbit. If the tank capacity is b and $b<a$ we can, starting from x, go anywhere *except* to points on the circle with radius $\|x\|$ and center at the origin. It is interesting to note in this last case that we can *approach* these exceptional places arbitrarily closely (in terms of the usual distance function!). Of course one refuelling makes all points accessible.

We enumerate below (without proof) a few other properties of the metric which are easily verified. For simplicity we consider only the case $n=2$, and define the area of a circle of radius r to be πr^2 and the area of a rectangle to be the product of its length and width. (i) The area of any circle with center at origin is πb^2. For other circles it is undefined. (ii) The area of a rectangle is a^2 or ab according as four or two of its vertices belong to some circle with center at origin. Otherwise the area is b^2. (iii) The area of a rectangle may be strictly smaller than that of a subrectangle. (iv) The area of a union of disjoint rectangles is the sum of its component rectangles. Here two rectangles with a common edge are not considered disjoint.

Reference

1. S. K. Hildebrand and H. W. Milnes, An interesting metric space, this MAGAZINE, 41 (1968) 244–247.

ANSWERS

A472. If $\alpha \geqq \pi/4$ then $\beta \leqq \pi/4 \leqq \alpha$, and $2+\tan\alpha+\tan\beta \leqq 2+2\tan\alpha$. But $\sec\alpha = 1/\cos\alpha \geqq \sin\alpha/\cos\alpha = \tan\alpha$, and $\sec\beta \geqq 1$ so $2\sec\alpha+2\sec\beta \geqq 2\tan\alpha+2$. A similar proof holds for $\beta \geqq \pi/4$. Assume $\alpha \leqq \pi/4$ and $\beta \geqq \pi/4$. Then $2+\tan\alpha+\tan\beta \leqq 4 \leqq 2\sec\alpha+2\sec\beta$.

A473. No, it does not follow. Simply let $d(x, y)=\min(1, |x-y|)$, and note that this metric generates the usual topology in R, but $d(0, x)=d(0, y)=1$ if $1<x<y$.

A474. The area of the annular space is 144π which, to four decimal places, is $452\cdot 3893$. Thus 3893 452 represents *CHIC DEB*.

A475. Solution: $x^3-117y^3 \equiv 0, \pm 1 \pmod 9$ and hence cannot equal five.

A476. $z^2=z\cdot z$ implies that $z=\sqrt{z\cdot z}$ but $z=\sqrt{z\cdot z}$ implies $z=\sqrt{z\sqrt{z\cdot z}}$ and so on.

(Quickies on page 112.)

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